QUANTUM LATTICE MODEL

Reduced-Action Foundations and Planck-Unit Derivations

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Abstract

This paper develops the Planck-unit foundations of the Quantum Lattice Model (QLM) from the per-radian primitive triplet $\{\hbar, \ell_P, t_P\}$. In this framework the Planck scale is reformulated at the level of reduced action per radian rather than through dimensional square-root combinations of $\{G, \hbar, c\}$.

The QLM begins from the covariant primitive phase–flow law

$$E = \hbar \frac{d\theta}{d\tau},$$

where θ is dimensionless phase and τ is proper time along a worldline. Energy is therefore defined as reduced action flow per unit proper time.

The fundamental Planck energy arises as the saturated single-tick realization of this primitive relation:

$$\Delta\theta = 1, \qquad \Delta\tau = t_P, \qquad \Delta S = \hbar \quad \Rightarrow \quad E_P = \frac{\hbar}{t_P}.$$

The expression $E_P = \hbar/t_P$ is thus not an independent definition, but the single-tick realization of the per-radian phase–flow rule.

When expressed in terms of the primitive triplet $\{\hbar, \ell_P, t_P\}$, all Planck quantities reduce to algebraically minimal forms. Mechanical, electromagnetic, geometric, and thermodynamic units admit a unified representation within the same phase–action structure, with examples including

$$m_P = \frac{\hbar t_P}{\ell_P^2}, \qquad p_P = \frac{\hbar}{\ell_P}.$$

The Bohr identity

$$\hbar = m_e v_B a_0$$

is retained as a calibration bridge within the same reduced-action framework, showing that hydrogenic angular momentum connects algebraically to the same action quantum without introducing additional dimensional primitives.

Electromagnetic quantities—including the Planck charge, current, voltage, power, and impedance—emerge from the same primitive substitutions. In the per-radian normalization, the Planck impedance is

$$Z_P = \frac{Z_0}{4\pi},$$

with the corresponding loop-aggregated form

$$Z_P^{(\text{loop})} = 2\pi Z_P = \frac{Z_0}{2}, \qquad Z_P = \frac{Z_P^{(\text{loop})}}{2\pi}.$$

Through symbolic derivations, dimensional analyses, and CODATA-validated numerical evaluations, this work establishes the complete per-radian Planck-unit system on which the broader Quantum Lattice Model is constructed. It provides the formal Planck-scale reference layer for subsequent developments in gravitational impedance, invariant transport and Lorentz kinematics, routing admittance, and phase-coherent dynamical structure.

1 Introduction

Planck units are conventionally introduced as dimensional constructs formed from the constants $\{\hbar, G, c\}$, yielding expressions such as $\sqrt{\hbar c^5/G}$ and $\sqrt{\hbar c/G}$ that mix quantum, gravitational, and relativistic structure. In the Quantum Lattice Model (QLM), these square-root forms are not taken as primitive. Instead, all Planck quantities are organized as algebraic reductions of a per-radian phase–action structure built from

$$\{\hbar, \ell_\mathrm{P}, t_\mathrm{P}\}.$$

The foundational relation is the covariant phase–flow law

$$E = \hbar \frac{d\theta}{d\tau}, \tag{1.1}$$

where θ is dimensionless phase measured in radians and τ is proper time along a worldline. Energy is therefore reduced-action flow per unit proper time.

Angular frequency is defined by

$$\omega \equiv \frac{d\theta}{d\tau}, \tag{1.2}$$

so that Eq. (1.1) becomes the usual per-radian energy relation $E = \hbar\omega$. Cycle frequency is derived from angular frequency by $\nu = \omega/(2\pi)$. Thus full-cycle quantities such as $h = 2\pi\hbar$ are useful conversion forms, but they are not primitive in the QLM construction.

The Planck-scale realization of Eq. (1.1) occurs when one saturated lattice tick advances one radian of phase in one Planck proper-time interval:

$$\Delta\theta = 1, \qquad \Delta\tau = t_\mathrm{P}, \qquad \Delta S = \hbar. \tag{1.3}$$

The corresponding one-tick action throughput is

$$E_\mathrm{P} = \frac{\hbar}{t_\mathrm{P}}. \tag{1.4}$$

Thus $E_\mathrm{P} = \hbar/t_\mathrm{P}$ is not introduced as an independent definition. It is the saturated single-tick realization of the per-radian phase–flow law.

The primitive spatial and temporal increments are linked by the invariant lattice transport identity

$$c = \frac{\ell_\mathrm{P}}{t_\mathrm{P}}. \tag{1.5}$$

This identity fixes the causal update bound of the lattice: one coherent null phase hop advances one spatial lattice interval per proper-time tick. It also provides the bridge from temporal action throughput to spatial momentum throughput. In particular,

$$p_\mathrm{P} = \frac{\hbar}{\ell_\mathrm{P}}, \qquad \frac{E_\mathrm{P}}{p_\mathrm{P}} = \frac{\hbar/t_\mathrm{P}}{\hbar/\ell_\mathrm{P}} = \frac{\ell_\mathrm{P}}{t_\mathrm{P}} = c. \tag{1.6}$$

The reduced-action quantum cancels from the ratio, leaving the invariant transport speed.

From Eqs. (1.4) and (1.5), the Planck mass follows from the inertial relation $E = mc^2$:

$$m_{\mathrm{P}} = \frac{E_{\mathrm{P}}}{c^2} = \frac{\hbar}{t_{\mathrm{P}}}\left(\frac{t_{\mathrm{P}}}{\ell_{\mathrm{P}}}\right)^2 = \frac{\hbar t_{\mathrm{P}}}{\ell_{\mathrm{P}}^2}. \tag{1.7}$$

Thus the central QLM reduction chain is

$$\{\hbar, \ell_{\mathrm{P}}, t_{\mathrm{P}}\} \quad\Longrightarrow\quad c = \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}}, \qquad E_{\mathrm{P}} = \frac{\hbar}{t_{\mathrm{P}}}, \qquad p_{\mathrm{P}} = \frac{\hbar}{\ell_{\mathrm{P}}}, \qquad m_{\mathrm{P}} = \frac{\hbar t_{\mathrm{P}}}{\ell_{\mathrm{P}}^2}. \tag{1.8}$$

Primitive spacings and numerical convention. The primitive triplet

$$\{\hbar,\ \ell_{\mathrm{P}},\ t_{\mathrm{P}}\} \tag{1.9}$$

represents the reduced-action quantum, the Planck lattice spacing, and the fundamental proper-time tick. All Planck-sector quantities developed in this paper are required to reduce algebraically to this triplet and its derived combinations.

All numerical evaluations use CODATA 2022 together with the post-2019 SI, in which h, e, k_B, and c are defined constants. The electromagnetic constants ϵ_0, μ_0, and Z_0 inherit their uncertainties through the fine-structure constant α.

Calibration bridges. Although the primitive QLM structure is built from $\{\hbar, \ell_{\mathrm{P}}, t_{\mathrm{P}}\}$, two calibration bridges are useful for connecting the Planck-unit system to familiar atomic and electromagnetic quantities.

First, the Bohr identity

$$\hbar = m_e v_B a_0 \tag{1.10}$$

connects the reduced-action quantum to the mechanical angular momentum of the hydrogen ground state. This relation is not a new primitive; it is a hydrogenic calibration of the same reduced-action quantum.

Second, the per-radian impedance normalization

$$Z_{\mathrm{P}} = \frac{Z_0}{4\pi}, \qquad Y_{\mathrm{P}} = \frac{1}{Z_{\mathrm{P}}}, \tag{1.11}$$

organizes the electromagnetic Planck sector. With this normalization, the Planck charge may be written in the compact reduced-action form

$$q_{\mathrm{P}}^2 = \frac{\hbar}{Z_{\mathrm{P}}} = \hbar Y_{\mathrm{P}}. \tag{1.12}$$

The detailed voltage, current, power, impedance, and loop-aggregated relations are developed later in the electromagnetic Planck-unit section.

Scope of this work. This paper develops the complete Planck-unit foundation of the Quantum Lattice Model. Each canonical Planck quantity is first related to its traditional square-root expression and then shown to collapse to a minimal QLM representation in terms of

$$\{\hbar, \ell_{\mathrm{P}}, t_{\mathrm{P}}\}.$$

The emphasis is algebraic consistency across mechanical, electromagnetic, geometric, gravitational, thermodynamic, and hydrogenic calibration sectors.

Later papers in the QLM series use this reference layer to develop the Planck energy-density cap, Lorentz kinematics from invariant proper ticks, routing admittance and gravitational throttling, and quantum phase transport. Those developments rely on the primitive reductions established here, but their dynamical content is not rederived in the present paper.

2 Constants and Conventions (CODATA 2022)

Table 1: Exact SI-defined constants and CODATA 2022 recommended values used throughout this paper. The Quantum Lattice Model treats the triplet $(\hbar, \ell_\mathrm{P}, t_\mathrm{P})$ as the primitive reduced-action lattice set, from which other Planck quantities follow algebraically. Numerical values follow [12].

Quantity	Symbol	Value (CODATA 2022)
Exact SI-defined constants		
Speed of light	c	$2.997\,925 \times 10^{8}\,\mathrm{m\,s^{-1}}$
Planck constant	h	$6.626\,070 \times 10^{-34}\,\mathrm{J\,s}$
Reduced Planck constant	$\hbar$	$1.054\,572 \times 10^{-34}\,\mathrm{J\,s}$
Elementary charge	e	$1.602\,177 \times 10^{-19}\,\mathrm{C}$
Boltzmann constant	k_B	$1.380\,649 \times 10^{-23}\,\mathrm{J\,K^{-1}}$
Measured CODATA constants		
Fine-structure constant	α	$7.297\,353 \times 10^{-3}$
Electron mass	m_e	$9.109\,384 \times 10^{-31}\,\mathrm{kg}$
Bohr radius	a_0	$5.291\,772 \times 10^{-11}\,\mathrm{m}$
Vacuum impedance	Z_0	$3.767\,303 \times 10^{2}\,\Omega$
Vacuum permeability	μ_0	$1.256\,637 \times 10^{-6}\,\mathrm{N\,A^{-2}}$
Vacuum permittivity	ϵ_0	$8.854\,188 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Newtonian gravitational constant	G	$6.674\,300 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
QLM primitive lattice scales		
Planck time	t_P	$5.391\,247 \times 10^{-44}\,\mathrm{s}$
Planck length	ℓ_P	$1.616\,255 \times 10^{-35}\,\mathrm{m}$
Derived QLM Planck quantities		
Planck angular frequency	$\omega_P = 1/t_\mathrm{P}$	$1.854\,858 \times 10^{43}\,\mathrm{s^{-1}}$
Planck energy	$E_\mathrm{P} = \hbar/t_\mathrm{P}$	$1.956\,081 \times 10^{9}\,\mathrm{J}$
Planck mass	$m_\mathrm{P} = \hbar t_\mathrm{P}/\ell_\mathrm{P}^2$	$2.176\,434 \times 10^{-8}\,\mathrm{kg}$

3 Primitive Reduced-Action Structure

The Quantum Lattice Model begins from the reduced-action phase relation

$$dS = \hbar\, d\theta, \tag{3.1}$$

which states that reduced action is transported in direct proportion to phase advance measured in radians. The phase variable θ is dimensionless, so one radian of coherent phase advance carries one quantum of reduced action $\hbar$.

Dividing Eq. (3.1) by proper time τ along a worldline gives

$$E \equiv \frac{dS}{d\tau} = \hbar \frac{d\theta}{d\tau}. \tag{3.2}$$

This is the covariant phase–flow law of the QLM. Energy is therefore reduced-action flow per unit proper time.

Defining angular frequency by

$$\omega \equiv \frac{d\theta}{d\tau}, \tag{3.3}$$

Eq. (3.2) becomes

$$E = \hbar\omega. \tag{3.4}$$

The corresponding cycle frequency is derived,

$$\nu = \frac{\omega}{2\pi}, \tag{3.5}$$

and the full-cycle action constant is

$$h = 2\pi\hbar. \tag{3.6}$$

Thus the per-radian quantity $\hbar$ is primitive in the QLM construction, while the full-cycle quantity h is a derived conversion form.

For a single saturated lattice tick,

$$\Delta\theta = 1, \qquad \Delta\tau = t_{\mathrm{P}}, \tag{3.7}$$

so that

$$\Delta S = \hbar\,\Delta\theta = \hbar. \tag{3.8}$$

The corresponding one-tick reduced-action throughput is

$$E_{\mathrm{P}} = \frac{\Delta S}{\Delta\tau} = \frac{\hbar}{t_{\mathrm{P}}}. \tag{3.9}$$

Therefore $E_{\mathrm{P}} = \hbar/t_{\mathrm{P}}$ is not introduced as an independent dimensional definition. It is the saturated single-tick realization of the phase–flow law.

The associated Planck angular frequency is

$$\omega_P = \frac{1}{t_{\mathrm{P}}}, \tag{3.10}$$

so the same result may be written

$$E_{\mathrm{P}} = \hbar\omega_P. \tag{3.11}$$

Equivalently, the reduced-action quantum can be recovered from the Planck energy and Planck proper-time tick:

$$\hbar = E_{\mathrm{P}}t_{\mathrm{P}} = \frac{E_{\mathrm{P}}}{\omega_P}. \tag{3.12}$$

The primitive action structure of the QLM is therefore summarized by

$$dS = \hbar\,d\theta, \qquad E = \hbar\frac{d\theta}{d\tau}, \qquad \Delta\theta = 1, \qquad \Delta\tau = t_{\mathrm{P}}, \qquad E_{\mathrm{P}} = \frac{\hbar}{t_{\mathrm{P}}}. \tag{3.13}$$

All subsequent Planck-unit reductions in this paper follow from this per-radian action structure together with the lattice transport identity $c = \ell_{\mathrm{P}}/t_{\mathrm{P}}$.

4 Planck Lattice Foundations

4.1 Planck Time

The Planck time t_{P} is the fundamental temporal increment of the Quantum Lattice Model (QLM). One saturated lattice tick corresponds to one radian of coherent phase evolution and transports one quantum of reduced action $\hbar$.

In conventional reduced-action Planck units, the Planck time is written as [1, 2]

$$t_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^5}}. \tag{4.1}$$

In the QLM this square-root expression is not taken as primitive. Instead, t_{P} is a primitive proper-time tick, and the spatial and temporal lattice increments are linked by the invariant transport identity

$$c = \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}}. \tag{4.2}$$

Solving Eq. (4.2) for the temporal increment gives the QLM lattice form

$$t_{\mathrm{P}} = \frac{\ell_{\mathrm{P}}}{c}. \tag{4.3}$$

Thus t_{P} is the temporal interval associated with one null lattice advance across one spatial Planck link ℓ_{P}.

Collapse of the conventional square-root form. To verify that the conventional expression (4.1) is compatible with the QLM lattice form (4.3), use the conventional reduced-action Planck-length identity

$$\ell_{\mathrm{P}}^2 = \frac{\hbar G}{c^3}. \tag{4.4}$$

Solving Eq. (4.4) for the product $\hbar G$ gives

$$\hbar G = \ell_{\mathrm{P}}^2 c^3. \tag{4.5}$$

Substituting Eq. (4.5) into Eq. (4.1) gives

$$\begin{aligned} t_{\mathrm{P}} &= \sqrt{\frac{\hbar G}{c^5}} \\ &= \sqrt{\frac{\ell_{\mathrm{P}}^2 c^3}{c^5}} \\ &= \sqrt{\frac{\ell_{\mathrm{P}}^2}{c^2}} \\ &= \frac{\ell_{\mathrm{P}}}{c}. \end{aligned} \tag{4.6}$$

Using the QLM transport identity $c = \ell_{\mathrm{P}}/t_{\mathrm{P}}$, the final expression becomes

$$t_{\mathrm{P}} = \frac{\ell_{\mathrm{P}}}{c} = \frac{\ell_{\mathrm{P}}}{\ell_{\mathrm{P}}/t_{\mathrm{P}}} = t_{\mathrm{P}}. \tag{4.7}$$

Therefore the conventional square-root expression collapses exactly to the QLM primitive temporal tick.

Dimensional check. The conventional expression has dimensions

$$\left[\sqrt{\frac{\hbar G}{c^5}}\right] = \sqrt{\frac{(\mathrm{kg\,m^2\,s^{-1}})(\mathrm{m^3\,kg^{-1}\,s^{-2}})}{(\mathrm{m\,s^{-1}})^5}}$$
$$= \sqrt{\frac{\mathrm{m^5\,s^{-3}}}{\mathrm{m^5\,s^{-5}}}}$$
$$= \sqrt{\mathrm{s}^2} = \mathrm{s}. \tag{4.8}$$

The QLM form $t_\mathrm{P} = \ell_\mathrm{P}/c$ has the same dimension,

$$\left[\frac{\ell_\mathrm{P}}{c}\right] = \frac{\mathrm{m}}{\mathrm{m\,s^{-1}}} = \mathrm{s}. \tag{4.9}$$

Numerical evaluation. Using CODATA 2022 values,

$$t_\mathrm{P} = \sqrt{\frac{(1.054\,572\times10^{-34}\,\mathrm{J\,s})(6.674\,300\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})}{(2.997\,925\times10^{8}\,\mathrm{m\,s^{-1}})^5}}$$
$$= 5.391\,247\times10^{-44}\,\mathrm{s}. \tag{4.10}$$

The QLM lattice form gives the same value:

$$t_\mathrm{P} = \frac{\ell_\mathrm{P}}{c}$$
$$= \frac{1.616\,255\times10^{-35}\,\mathrm{m}}{2.997\,925\times10^{8}\,\mathrm{m\,s^{-1}}}$$
$$= 5.391\,247\times10^{-44}\,\mathrm{s}. \tag{4.11}$$

Thus,

$$t_\mathrm{P} = \sqrt{\frac{\hbar G}{c^5}} = \frac{\ell_\mathrm{P}}{c} \tag{4.12}$$

and the conventional Planck-time expression is recovered as a derived identity rather than as a primitive starting point.

4.2 Planck Length

The Planck length ℓ_P is the fundamental spatial increment of the Quantum Lattice Model. One null lattice update advances one spatial Planck link in one Planck proper-time tick.

In conventional reduced-action Planck units, the Planck length is written as [1, 2]

$$\ell_\mathrm{P} = \sqrt{\frac{\hbar G}{c^3}}. \tag{4.13}$$

In the QLM this square-root expression is not taken as primitive. Instead, ℓ_P is a primitive lattice spacing linked to the Planck proper-time tick by the invariant transport identity

$$c = \frac{\ell_\mathrm{P}}{t_\mathrm{P}}. \tag{4.14}$$

Solving Eq. (4.14) for the spatial increment gives the QLM lattice form

$$\ell_\mathrm{P} = c\,t_\mathrm{P}. \tag{4.15}$$

Thus ℓ_P is the spatial interval associated with one null lattice advance during one Planck proper-time tick.

Collapse of the conventional square-root form. To verify that the conventional expression (4.13) is compatible with the QLM lattice form (4.15), use the Planck-time identity established in the previous subsection,

$$t_{\mathrm{P}} = \sqrt{\frac{\hbar G}{c^5}}. \tag{4.16}$$

Squaring Eq. (4.16) gives

$$t_{\mathrm{P}}^2 = \frac{\hbar G}{c^5}. \tag{4.17}$$

Solving for the product $\hbar G$ gives

$$\hbar G = t_{\mathrm{P}}^2 c^5. \tag{4.18}$$

Substituting Eq. (4.18) into Eq. (4.13) gives

$$\begin{aligned} \ell_{\mathrm{P}} &= \sqrt{\frac{\hbar G}{c^3}} \\ &= \sqrt{\frac{t_{\mathrm{P}}^2 c^5}{c^3}} \\ &= \sqrt{t_{\mathrm{P}}^2 c^2} \\ &= c\,t_{\mathrm{P}}. \end{aligned} \tag{4.19}$$

Using the QLM transport identity $c = \ell_{\mathrm{P}}/t_{\mathrm{P}}$, the final expression becomes

$$\ell_{\mathrm{P}} = c\,t_{\mathrm{P}} = \left(\frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}}\right) t_{\mathrm{P}} = \ell_{\mathrm{P}}. \tag{4.20}$$

Therefore the conventional square-root expression collapses exactly to the QLM primitive spatial lattice spacing.

Dimensional check. The conventional expression has dimensions

$$\begin{aligned} \left[\sqrt{\frac{\hbar G}{c^3}}\right] &= \sqrt{\frac{(\mathrm{kg\,m^2\,s^{-1}})(\mathrm{m^3\,kg^{-1}\,s^{-2}})}{(\mathrm{m\,s^{-1}})^3}} \\ &= \sqrt{\frac{\mathrm{m^5\,s^{-3}}}{\mathrm{m^3\,s^{-3}}}} \\ &= \sqrt{\mathrm{m^2}} = \mathrm{m}. \end{aligned} \tag{4.21}$$

The QLM form $\ell_{\mathrm{P}} = ct_{\mathrm{P}}$ has the same dimension,

$$[ct_{\mathrm{P}}] = (\mathrm{m\,s^{-1}})(\mathrm{s}) = \mathrm{m}. \tag{4.22}$$

Numerical evaluation. Using CODATA 2022 values,

$$\begin{aligned} \ell_{\mathrm{P}} &= \sqrt{\frac{(1.054\,572 \times 10^{-34}\,\mathrm{J\,s})(6.674\,300 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})}{(2.997\,925 \times 10^{8}\,\mathrm{m\,s^{-1}})^3}} \\ &= 1.616\,255 \times 10^{-35}\,\mathrm{m}. \end{aligned} \tag{4.23}$$

The QLM lattice form gives the same value:

$$\begin{aligned}\ell_\mathrm{P} &= c\,t_\mathrm{P} \\ &= (2.997\,925 \times 10^{8}\,\mathrm{m\,s^{-1}})(5.391\,247 \times 10^{-44}\,\mathrm{s}) \\ &= 1.616\,255 \times 10^{-35}\,\mathrm{m}. \end{aligned} \tag{4.24}$$

Thus,

$$\ell_\mathrm{P} = \sqrt{\frac{\hbar G}{c^3}} = c\,t_\mathrm{P} \tag{4.25}$$

and the conventional Planck-length expression is recovered as a derived identity rather than as a primitive starting point.

4.3 Planck Energy

In the Quantum Lattice Model, the Planck energy E_P is not introduced as a dimensional square-root primitive. It arises directly from the covariant phase–flow law,

$$E = \hbar \frac{d\theta}{d\tau}, \tag{4.26}$$

where θ is dimensionless phase and τ is proper time along a worldline.

For a single saturated lattice tick,

$$\Delta\theta = 1, \qquad \Delta\tau = t_\mathrm{P}, \qquad \Delta S = \hbar, \tag{4.27}$$

so the corresponding one-tick reduced-action throughput is

$$E_\mathrm{P} = \frac{\Delta S}{\Delta\tau} = \frac{\hbar}{t_\mathrm{P}}. \tag{4.28}$$

Thus E_P is the saturated single-radian phase-action throughput permitted by one Planck proper-time tick.

In conventional reduced-action Planck units, the same quantity is written as [1, 2]

$$E_\mathrm{P} = \sqrt{\frac{\hbar c^5}{G}}. \tag{4.29}$$

The purpose of this subsection is to show that this conventional square-root form collapses exactly to the QLM identity $E_\mathrm{P} = \hbar/t_\mathrm{P}$.

Collapse of the conventional square-root form. Use the Planck-length identity

$$\ell_\mathrm{P}^2 = \frac{\hbar G}{c^3}. \tag{4.30}$$

Solving Eq. (4.30) for G gives

$$G = \frac{\ell_\mathrm{P}^2 c^3}{\hbar}. \tag{4.31}$$

Substituting Eq. (4.31) into Eq. (4.29) yields

$$\begin{aligned} E_{\mathrm{P}} &= \sqrt{\frac{\hbar c^5}{\ell_{\mathrm{P}}^2 c^3/\hbar}} \\ &= \sqrt{\frac{\hbar^2 c^2}{\ell_{\mathrm{P}}^2}} \\ &= \frac{\hbar c}{\ell_{\mathrm{P}}}. \end{aligned} \tag{4.32}$$

Using the QLM lattice transport identity

$$c = \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}}, \tag{4.33}$$

Eq. (4.32) becomes

$$\begin{aligned} E_{\mathrm{P}} &= \frac{\hbar}{\ell_{\mathrm{P}}} \left(\frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}} \right) \\ &= \frac{\hbar}{t_{\mathrm{P}}}. \end{aligned} \tag{4.34}$$

Therefore the conventional Planck-energy expression collapses exactly to the QLM one-tick reduced-action throughput.

Dimensional check. The conventional square-root expression has dimensions

$$\begin{aligned} \left[\sqrt{\frac{\hbar c^5}{G}} \right] &= \sqrt{\frac{(\mathrm{kg\,m^2\,s^{-1}})(\mathrm{m\,s^{-1}})^5}{\mathrm{m^3\,kg^{-1}\,s^{-2}}}} \\ &= \sqrt{\frac{\mathrm{kg\,m^2\,s^{-1}\,m^5\,s^{-5}}}{\mathrm{m^3\,kg^{-1}\,s^{-2}}}} \\ &= \sqrt{\mathrm{kg^2\,m^4\,s^{-4}}} \\ &= \mathrm{kg\,m^2\,s^{-2}} = \mathrm{J}. \end{aligned} \tag{4.35}$$

The QLM form has the same dimension,

$$\left[\frac{\hbar}{t_{\mathrm{P}}} \right] = \frac{\mathrm{J\,s}}{\mathrm{s}} = \mathrm{J}. \tag{4.36}$$

Numerical evaluation. Using CODATA 2022 values,

$$\begin{aligned} E_{\mathrm{P}} &= \sqrt{\frac{(1.054\,572 \times 10^{-34}\,\mathrm{J\,s})(2.997\,925 \times 10^{8}\,\mathrm{m\,s^{-1}})^5}{6.674\,300 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}}} \\ &= 1.956\,081 \times 10^{9}\,\mathrm{J}. \end{aligned} \tag{4.37}$$

The QLM one-tick form gives the same value:

$$\begin{aligned} E_{\mathrm{P}} &= \frac{\hbar}{t_{\mathrm{P}}} \\ &= \frac{1.054\,572 \times 10^{-34}\,\mathrm{J\,s}}{5.391\,247 \times 10^{-44}\,\mathrm{s}} \\ &= 1.956\,081 \times 10^{9}\,\mathrm{J}. \end{aligned} \tag{4.38}$$

Thus,

$$E_{\mathrm{P}} = \sqrt{\frac{\hbar c^5}{G}} = \frac{\hbar c}{\ell_{\mathrm{P}}} = \frac{\hbar}{t_{\mathrm{P}}} \tag{4.39}$$

and the conventional Planck-energy expression is recovered as a derived identity rather than as a primitive starting point.

Energy–mass closure. The relativistic identity $E = mc^2$ [6] is consistent with the QLM Planck energy when applied to the Planck mass. Using the QLM Planck-mass form

$$m_{\mathrm{P}} = \frac{\hbar t_{\mathrm{P}}}{\ell_{\mathrm{P}}^2}, \tag{4.40}$$

and the lattice transport identity

$$c = \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}}, \tag{4.41}$$

the relativistic energy becomes

$$\begin{aligned} m_{\mathrm{P}} c^2 &= \frac{\hbar t_{\mathrm{P}}}{\ell_{\mathrm{P}}^2} \left(\frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}}\right)^2 \\ &= \frac{\hbar}{t_{\mathrm{P}}} = E_{\mathrm{P}}. \end{aligned} \tag{4.42}$$

Thus the mass–energy relation reproduces the same QLM per-radian Planck-energy identity.

4.4 Planck Angular Frequency

In the Quantum Lattice Model, each saturated lattice tick advances the coherent phase by exactly one radian. Since the primitive proper-time tick is t_{P}, the corresponding Planck angular frequency is

$$\omega_P = \frac{1}{t_{\mathrm{P}}}. \tag{4.43}$$

This is the fundamental temporal phase rate associated with one Planck-scale update. Its units are radians per second, with radians dimensionless.

The corresponding cycle frequency is derived by dividing the angular frequency by 2π:

$$\nu_P = \frac{\omega_P}{2\pi} = \frac{1}{2\pi t_{\mathrm{P}}}. \tag{4.44}$$

Thus angular frequency is the natural QLM rate variable, while cycle frequency is a derived conversion form.

Energy–frequency correspondence. From the primitive phase–flow law

$$E = \hbar \frac{d\theta}{d\tau}, \tag{4.45}$$

and the definition

$$\omega \equiv \frac{d\theta}{d\tau}, \tag{4.46}$$

energy is related to angular frequency by

$$E = \hbar\omega. \tag{4.47}$$

For a saturated lattice tick,

$$E_{\mathrm{P}} = \hbar\omega_P = \frac{\hbar}{t_{\mathrm{P}}}. \tag{4.48}$$

This recovers the same one-tick Planck energy derived in the previous subsection.

Collapse of the conventional square-root form. The conventional reduced-action angular frequency corresponding to Planck energy is

$$\omega_P = \sqrt{\frac{c^5}{\hbar G}}, \tag{4.49}$$

which follows from $E_{\mathrm{P}} = \hbar\omega_P$ together with the conventional Planck-energy expression [1, 2].

To verify compatibility with the QLM expression (4.43), use the Planck-length identity

$$\ell_{\mathrm{P}}^2 = \frac{\hbar G}{c^3}. \tag{4.50}$$

Solving Eq. (4.50) for $\hbar G$ gives

$$\hbar G = \ell_{\mathrm{P}}^2 c^3. \tag{4.51}$$

Substituting Eq. (4.51) into Eq. (4.49) gives

$$\begin{aligned} \omega_P &= \sqrt{\frac{c^5}{\hbar G}} \\ &= \sqrt{\frac{c^5}{\ell_{\mathrm{P}}^2 c^3}} \\ &= \sqrt{\frac{c^2}{\ell_{\mathrm{P}}^2}} \\ &= \frac{c}{\ell_{\mathrm{P}}}. \end{aligned} \tag{4.52}$$

Using the lattice transport identity

$$c = \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}}, \tag{4.53}$$

Eq. (4.52) becomes

$$\begin{aligned} \omega_P &= \frac{1}{\ell_{\mathrm{P}}}\left(\frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}}\right) \\ &= \frac{1}{t_{\mathrm{P}}}. \end{aligned} \tag{4.54}$$

Therefore the conventional square-root expression collapses exactly to the QLM Planck angular frequency.

Dimensional check. The conventional square-root expression has dimensions

$$\begin{aligned} \left[\sqrt{\frac{c^5}{\hbar G}}\right] &= \sqrt{\frac{(\mathrm{m\,s^{-1}})^5}{(\mathrm{kg\,m^2\,s^{-1}})(\mathrm{m^3\,kg^{-1}\,s^{-2}})}} \\ &= \sqrt{\frac{\mathrm{m^5\,s^{-5}}}{\mathrm{m^5\,s^{-3}}}} \\ &= \sqrt{\mathrm{s^{-2}}} = \mathrm{s^{-1}}. \end{aligned} \tag{4.55}$$

The QLM form has the same dimension,

$$\left[\frac{1}{t_\mathrm{P}}\right] = \mathrm{s}^{-1}. \tag{4.56}$$

Numerical evaluation. Using CODATA 2022 values,

$$\begin{aligned}\omega_P &= \sqrt{\frac{(2.997\,925 \times 10^{8}\,\mathrm{m\,s^{-1}})^5}{(1.054\,572 \times 10^{-34}\,\mathrm{J\,s})(6.674\,300 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})}}\\ &= 1.854\,858 \times 10^{43}\,\mathrm{s}^{-1}.\end{aligned} \tag{4.57}$$

The QLM tick-rate form gives the same value:

$$\begin{aligned}\omega_P &= \frac{1}{t_\mathrm{P}}\\ &= \frac{1}{5.391\,247 \times 10^{-44}\,\mathrm{s}}\\ &= 1.854\,858 \times 10^{43}\,\mathrm{s}^{-1}.\end{aligned} \tag{4.58}$$

The derived cycle frequency is

$$\nu_P = \frac{\omega_P}{2\pi} = 2.953\,250 \times 10^{42}\,\mathrm{Hz}. \tag{4.59}$$

Thus,

$$\omega_P = \sqrt{\frac{c^5}{\hbar G}} = \frac{c}{\ell_\mathrm{P}} = \frac{1}{t_\mathrm{P}} \tag{4.60}$$

and the conventional Planck angular-frequency expression is recovered as a derived identity rather than as a primitive starting point.

4.5 Planck Wavenumber

In the Quantum Lattice Model, the Planck angular wavenumber k_P is the spatial phase rate associated with one radian of phase advance across one Planck lattice spacing. Since the primitive spatial increment is ℓ_P, the corresponding Planck angular wavenumber is

$$k_P = \frac{1}{\ell_\mathrm{P}}. \tag{4.61}$$

Its units are radians per meter, with radians dimensionless.

This is the spatial analogue of the Planck angular frequency,

$$\omega_P = \frac{1}{t_\mathrm{P}}. \tag{4.62}$$

No additional 2π factor is required: both ω and k are defined per radian. Cycle frequency and ordinary wavelength forms are derived conversion forms.

Wave–frequency closure. Using the invariant lattice transport identity

$$c = \frac{\ell_\mathrm{P}}{t_\mathrm{P}}, \tag{4.63}$$

the ratio of angular frequency to angular wavenumber becomes

$$\begin{aligned}\frac{\omega_P}{k_P} &= \frac{1/t_\mathrm{P}}{1/\ell_\mathrm{P}} \\ &= \frac{\ell_\mathrm{P}}{t_\mathrm{P}} \\ &= c.\end{aligned} \tag{4.64}$$

Thus the fundamental QLM null phase-transport relation is

$$\frac{\omega_P}{k_P} = c. \tag{4.65}$$

This is the angular-variable form of the null dispersion relation.

Collapse of the conventional square-root form. Because the conventional Planck length is

$$\ell_\mathrm{P} = \sqrt{\frac{\hbar G}{c^3}}, \tag{4.66}$$

the corresponding inverse spatial phase scale is

$$\begin{aligned}k_P &= \frac{1}{\ell_\mathrm{P}} \\ &= \frac{1}{\sqrt{\hbar G/c^3}} \\ &= \sqrt{\frac{c^3}{\hbar G}}.\end{aligned} \tag{4.67}$$

To verify compatibility with the QLM primitive form, use the Planck-length identity

$$\ell_\mathrm{P}^2 = \frac{\hbar G}{c^3}. \tag{4.68}$$

Solving for $\hbar G$ gives

$$\hbar G = \ell_\mathrm{P}^2 c^3. \tag{4.69}$$

Substituting Eq. (4.69) into Eq. (4.67) gives

$$\begin{aligned}k_P &= \sqrt{\frac{c^3}{\ell_\mathrm{P}^2 c^3}} \\ &= \sqrt{\frac{1}{\ell_\mathrm{P}^2}} \\ &= \frac{1}{\ell_\mathrm{P}}.\end{aligned} \tag{4.70}$$

Therefore the conventional inverse-Planck-length expression collapses exactly to the QLM spatial phase-rate identity.

Dimensional check. The square-root expression has dimensions

$$\left[\sqrt{\frac{c^3}{\hbar G}}\right] = \sqrt{\frac{(\mathrm{m\,s^{-1}})^3}{(\mathrm{kg\,m^2\,s^{-1}})(\mathrm{m^3\,kg^{-1}\,s^{-2}})}}$$
$$= \sqrt{\frac{\mathrm{m^3\,s^{-3}}}{\mathrm{m^5\,s^{-3}}}}$$
$$= \sqrt{\mathrm{m^{-2}}} = \mathrm{m^{-1}}. \tag{4.71}$$

The QLM form has the same dimension,

$$\left[\frac{1}{\ell_{\mathrm{P}}}\right] = \mathrm{m^{-1}}. \tag{4.72}$$

The wave-speed ratio also has the correct dimension,

$$\left[\frac{\omega_P}{k_P}\right] = \frac{\mathrm{s^{-1}}}{\mathrm{m^{-1}}} = \mathrm{m\,s^{-1}}. \tag{4.73}$$

Numerical evaluation. Using CODATA 2022 values,

$$k_P = \sqrt{\frac{(2.997\,925\times10^{8}\,\mathrm{m\,s^{-1}})^3}{(1.054\,572\times10^{-34}\,\mathrm{J\,s})(6.674\,300\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})}}$$
$$= 6.187\,066\times10^{34}\,\mathrm{m^{-1}}. \tag{4.74}$$

The QLM lattice form gives the same value:

$$k_P = \frac{1}{\ell_{\mathrm{P}}}$$
$$= \frac{1}{1.616\,255\times10^{-35}\,\mathrm{m}}$$
$$= 6.187\,066\times10^{34}\,\mathrm{m^{-1}}. \tag{4.75}$$

Using the Planck angular frequency from the previous subsection,

$$\frac{\omega_P}{k_P} = \frac{1.854\,858\times10^{43}\,\mathrm{s^{-1}}}{6.187\,066\times10^{34}\,\mathrm{m^{-1}}}$$
$$= 2.997\,925\times10^{8}\,\mathrm{m\,s^{-1}} = c. \tag{4.76}$$

Thus,

$$k_P = \sqrt{\frac{c^3}{\hbar G}} = \frac{1}{\ell_{\mathrm{P}}}, \qquad \frac{\omega_P}{k_P} = c \tag{4.77}$$

and the Planck angular wavenumber is recovered as the inverse spatial phase scale of the QLM lattice.

4.6 Planck Mass

In the Quantum Lattice Model, the Planck mass m_{P} follows from the saturated Planck energy together with the relativistic inertial relation $E = mc^2$ [6]. Since

$$E_{\mathrm{P}} = \frac{\hbar}{t_{\mathrm{P}}}, \tag{4.78}$$

the corresponding mass scale is

$$m_{\mathrm{P}} = \frac{E_{\mathrm{P}}}{c^2} = \frac{\hbar}{t_{\mathrm{P}} c^2}. \tag{4.79}$$

Using the lattice transport identity

$$c = \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}}, \tag{4.80}$$

Eq. (4.79) becomes

$$\begin{aligned} m_{\mathrm{P}} &= \frac{\hbar}{t_{\mathrm{P}}} \left(\frac{t_{\mathrm{P}}}{\ell_{\mathrm{P}}}\right)^2 \\ &= \frac{\hbar t_{\mathrm{P}}}{\ell_{\mathrm{P}}^2}. \end{aligned} \tag{4.81}$$

This is the QLM lattice form of the Planck mass.

In conventional reduced-action Planck units, the same mass scale is written as [1, 2]

$$m_{\mathrm{P}} = \sqrt{\frac{\hbar c}{G}}. \tag{4.82}$$

The purpose of the following reduction is to show that this square-root form collapses exactly to Eq. (4.81).

Collapse of the conventional square-root form. Use the Planck-length identity

$$\ell_{\mathrm{P}}^2 = \frac{\hbar G}{c^3}. \tag{4.83}$$

Solving Eq. (4.83) for G gives

$$G = \frac{\ell_{\mathrm{P}}^2 c^3}{\hbar}. \tag{4.84}$$

Substituting Eq. (4.84) into Eq. (4.82) gives

$$\begin{aligned} m_{\mathrm{P}} &= \sqrt{\frac{\hbar c}{\ell_{\mathrm{P}}^2 c^3/\hbar}} \\ &= \sqrt{\frac{\hbar^2}{\ell_{\mathrm{P}}^2 c^2}} \\ &= \frac{\hbar}{\ell_{\mathrm{P}} c}. \end{aligned} \tag{4.85}$$

Using the lattice transport identity $c = \ell_{\mathrm{P}}/t_{\mathrm{P}}$, this becomes

$$\begin{aligned} m_{\mathrm{P}} &= \frac{\hbar}{\ell_{\mathrm{P}}(\ell_{\mathrm{P}}/t_{\mathrm{P}})} \\ &= \frac{\hbar t_{\mathrm{P}}}{\ell_{\mathrm{P}}^2}, \end{aligned} \tag{4.86}$$

matching the QLM lattice form.

Dimensional check. The conventional square-root expression has dimensions

$$\left[\sqrt{\frac{\hbar c}{G}}\right] = \sqrt{\frac{(\mathrm{kg\,m^2\,s^{-1}})(\mathrm{m\,s^{-1}})}{\mathrm{m^3\,kg^{-1}\,s^{-2}}}} = \sqrt{\frac{\mathrm{kg\,m^3\,s^{-2}}}{\mathrm{m^3\,kg^{-1}\,s^{-2}}}} = \sqrt{\mathrm{kg^2}} = \mathrm{kg}. \tag{4.87}$$

The QLM form has the same dimension,

$$\left[\frac{\hbar t_\mathrm{P}}{\ell_\mathrm{P}^2}\right] = \frac{(\mathrm{kg\,m^2\,s^{-1}})(\mathrm{s})}{\mathrm{m^2}} = \mathrm{kg}. \tag{4.88}$$

Numerical evaluation. Using CODATA 2022 values,

$$m_\mathrm{P} = \sqrt{\frac{(1.054\,572 \times 10^{-34}\,\mathrm{J\,s})(2.997\,925 \times 10^{8}\,\mathrm{m\,s^{-1}})}{6.674\,300 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}}} = 2.176\,434 \times 10^{-8}\,\mathrm{kg}. \tag{4.89}$$

The QLM lattice form gives the same value:

$$m_\mathrm{P} = \frac{\hbar t_\mathrm{P}}{\ell_\mathrm{P}^2} = \frac{(1.054\,572 \times 10^{-34}\,\mathrm{J\,s})(5.391\,247 \times 10^{-44}\,\mathrm{s})}{(1.616\,255 \times 10^{-35}\,\mathrm{m})^2} = 2.176\,434 \times 10^{-8}\,\mathrm{kg}. \tag{4.90}$$

Thus,

$$m_\mathrm{P} = \sqrt{\frac{\hbar c}{G}} = \frac{\hbar}{\ell_\mathrm{P} c} = \frac{\hbar t_\mathrm{P}}{\ell_\mathrm{P}^2}. \tag{4.91}$$

The conventional Planck-mass expression is therefore recovered as a derived identity rather than as a primitive starting point.

Consistency with $E = mc^2$. Using $m_\mathrm{P} = \hbar t_\mathrm{P}/\ell_\mathrm{P}^2$ and $c = \ell_\mathrm{P}/t_\mathrm{P}$,

$$m_\mathrm{P} c^2 = \left(\frac{\hbar t_\mathrm{P}}{\ell_\mathrm{P}^2}\right)\left(\frac{\ell_\mathrm{P}}{t_\mathrm{P}}\right)^2 = \frac{\hbar}{t_\mathrm{P}} = E_\mathrm{P}. \tag{4.92}$$

Thus the relativistic mass–energy relation reproduces the saturated QLM Planck energy.

4.7 Planck Momentum

In the Quantum Lattice Model, the Planck momentum p_P follows from the saturated Planck energy and the invariant lattice transport speed. For null propagation,

$$p_\mathrm{P} = \frac{E_\mathrm{P}}{c}, \tag{4.93}$$

with

$$E_\mathrm{P} = \frac{\hbar}{t_\mathrm{P}}. \tag{4.94}$$

Using the lattice transport identity

$$c = \frac{\ell_\mathrm{P}}{t_\mathrm{P}}, \tag{4.95}$$

Eq. (4.93) becomes

$$\begin{aligned} p_\mathrm{P} &= \frac{\hbar/t_\mathrm{P}}{\ell_\mathrm{P}/t_\mathrm{P}} \\ &= \frac{\hbar}{\ell_\mathrm{P}}. \end{aligned} \tag{4.96}$$

This is the QLM lattice form of the Planck momentum. It is the spatial dual of the Planck energy relation $E_\mathrm{P} = \hbar/t_\mathrm{P}$.

In conventional reduced-action Planck units, the same momentum scale is written as [1, 2]

$$p_\mathrm{P} = \sqrt{\frac{\hbar c^3}{G}}. \tag{4.97}$$

The purpose of the following reduction is to show that this square-root form collapses exactly to Eq. (4.96).

Collapse of the conventional square-root form. Use the Planck-length identity

$$\ell_\mathrm{P}^2 = \frac{\hbar G}{c^3}. \tag{4.98}$$

Solving Eq. (4.98) for G gives

$$G = \frac{\ell_\mathrm{P}^2 c^3}{\hbar}. \tag{4.99}$$

Substituting Eq. (4.99) into Eq. (4.97) gives

$$\begin{aligned} p_\mathrm{P} &= \sqrt{\frac{\hbar c^3}{\ell_\mathrm{P}^2 c^3/\hbar}} \\ &= \sqrt{\frac{\hbar^2}{\ell_\mathrm{P}^2}} \\ &= \frac{\hbar}{\ell_\mathrm{P}}. \end{aligned} \tag{4.100}$$

Therefore the conventional Planck-momentum expression collapses exactly to the QLM spatial action-gradient form.

Wavenumber form. The Planck angular wavenumber is

$$k_P = \frac{1}{\ell_{\mathrm{P}}}. \tag{4.101}$$

Since QLM uses the spatial phase–momentum relation

$$p = \hbar k, \tag{4.102}$$

the Planck momentum may also be written

$$p_{\mathrm{P}} = \hbar k_P = \frac{\hbar}{\ell_{\mathrm{P}}}. \tag{4.103}$$

Together with $E_{\mathrm{P}} = \hbar/t_{\mathrm{P}}$, this gives the null transport bridge

$$\begin{aligned} \frac{E_{\mathrm{P}}}{p_{\mathrm{P}}} &= \frac{\hbar/t_{\mathrm{P}}}{\hbar/\ell_{\mathrm{P}}} \\ &= \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}} \\ &= c. \end{aligned} \tag{4.104}$$

Thus the reduced-action quantum cancels from the ratio, leaving the invariant lattice speed.

Dimensional check. The conventional square-root expression has dimensions

$$\begin{aligned} \left[\sqrt{\frac{\hbar c^3}{G}}\right] &= \sqrt{\frac{(\mathrm{kg\,m^2\,s^{-1}})(\mathrm{m\,s^{-1}})^3}{\mathrm{m^3\,kg^{-1}\,s^{-2}}}} \\ &= \sqrt{\frac{\mathrm{kg\,m^5\,s^{-4}}}{\mathrm{m^3\,kg^{-1}\,s^{-2}}}} \\ &= \sqrt{\mathrm{kg^2\,m^2\,s^{-2}}} \\ &= \mathrm{kg\,m\,s^{-1}}. \end{aligned} \tag{4.105}$$

The QLM form has the same dimension,

$$\begin{aligned} \left[\frac{\hbar}{\ell_{\mathrm{P}}}\right] &= \frac{\mathrm{kg\,m^2\,s^{-1}}}{\mathrm{m}} \\ &= \mathrm{kg\,m\,s^{-1}}. \end{aligned} \tag{4.106}$$

The wavenumber form also has the same dimension,

$$[\hbar k_P] = (\mathrm{kg\,m^2\,s^{-1}})(\mathrm{m^{-1}}) = \mathrm{kg\,m\,s^{-1}}. \tag{4.107}$$

Numerical evaluation. Using CODATA 2022 values,

$$\begin{aligned} p_{\mathrm{P}} &= \sqrt{\frac{(1.054\,572 \times 10^{-34}\,\mathrm{J\,s})(2.997\,925 \times 10^{8}\,\mathrm{m\,s^{-1}})^3}{6.674\,300 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}}} \\ &= 6.524\,785 \times 10^{1}\,\mathrm{kg\,m\,s^{-1}}. \end{aligned} \tag{4.108}$$

The QLM lattice form gives the same value:

$$\begin{aligned} p_{\mathrm{P}} &= \frac{\hbar}{\ell_{\mathrm{P}}} \\ &= \frac{1.054\,572 \times 10^{-34}\,\mathrm{J\,s}}{1.616\,255 \times 10^{-35}\,\mathrm{m}} \\ &= 6.524\,785 \times 10^{1}\,\mathrm{kg\,m\,s^{-1}}. \end{aligned} \tag{4.109}$$

Equivalently, using $k_P = 1/\ell_{\mathrm{P}}$,

$$\begin{aligned} p_{\mathrm{P}} &= \hbar k_P \\ &= \left(1.054\,572 \times 10^{-34}\,\mathrm{J\,s}\right)\left(6.187\,066 \times 10^{34}\,\mathrm{m^{-1}}\right) \\ &= 6.524\,785 \times 10^{1}\,\mathrm{kg\,m\,s^{-1}}. \end{aligned} \tag{4.110}$$

Finally,

$$\begin{aligned} \frac{E_{\mathrm{P}}}{c} &= \frac{1.956\,081 \times 10^{9}\,\mathrm{J}}{2.997\,925 \times 10^{8}\,\mathrm{m\,s^{-1}}} \\ &= 6.524\,785 \times 10^{1}\,\mathrm{kg\,m\,s^{-1}}, \end{aligned} \tag{4.111}$$

confirming agreement across the energy, spatial-gradient, and wavenumber forms.

Summary. The Planck momentum identities are

$$p_{\mathrm{P}} = \sqrt{\frac{\hbar c^3}{G}} = \frac{E_{\mathrm{P}}}{c} = \frac{\hbar}{\ell_{\mathrm{P}}} = \hbar k_P. \tag{4.112}$$

All expressions are algebraically identical and represent the same per-radian Planck momentum scale in the Quantum Lattice Model.

4.8 Planck Force

In the Quantum Lattice Model, the Planck force F_{P} follows from the saturated Planck energy as the spatial rate at which one Planck energy quantum is distributed across one lattice spacing. Since

$$E_{\mathrm{P}} = \frac{\hbar}{t_{\mathrm{P}}}, \tag{4.113}$$

the corresponding force scale is

$$F_{\mathrm{P}} = \frac{E_{\mathrm{P}}}{\ell_{\mathrm{P}}} = \frac{\hbar}{t_{\mathrm{P}}\ell_{\mathrm{P}}}. \tag{4.114}$$

This is the QLM lattice form of the Planck force.

Using the lattice transport identity

$$\ell_{\mathrm{P}} = c t_{\mathrm{P}}, \tag{4.115}$$

Eq. (4.114) may also be written

$$\begin{aligned} F_{\mathrm{P}} &= \frac{\hbar}{t_{\mathrm{P}}(c t_{\mathrm{P}})} \\ &= \frac{\hbar}{c t_{\mathrm{P}}^2}. \end{aligned} \tag{4.116}$$

Equivalently, because $p_\mathrm{P} = \hbar/\ell_\mathrm{P}$, the same force scale is the one-tick momentum-throughput scale

$$F_\mathrm{P} = \frac{p_\mathrm{P}}{t_\mathrm{P}} = \frac{\hbar}{\ell_\mathrm{P} t_\mathrm{P}}. \tag{4.117}$$

In conventional Planck units, the same force scale is written as

$$F_\mathrm{P} = \frac{c^4}{G}. \tag{4.118}$$

The purpose of the following reduction is to show that this expression collapses exactly to the QLM lattice form.

Collapse of the conventional form. Use the Planck-length identity

$$\ell_\mathrm{P}^2 = \frac{\hbar G}{c^3}. \tag{4.119}$$

Solving Eq. (4.119) for G gives

$$G = \frac{\ell_\mathrm{P}^2 c^3}{\hbar}. \tag{4.120}$$

Substituting Eq. (4.120) into Eq. (4.118) gives

$$\begin{aligned} F_\mathrm{P} &= \frac{c^4}{\ell_\mathrm{P}^2 c^3/\hbar} \\ &= \frac{\hbar c}{\ell_\mathrm{P}^2}. \end{aligned} \tag{4.121}$$

Using the lattice transport identity $c = \ell_\mathrm{P}/t_\mathrm{P}$, this becomes

$$\begin{aligned} F_\mathrm{P} &= \frac{\hbar}{\ell_\mathrm{P}^2}\left(\frac{\ell_\mathrm{P}}{t_\mathrm{P}}\right) \\ &= \frac{\hbar}{\ell_\mathrm{P} t_\mathrm{P}}. \end{aligned} \tag{4.122}$$

Therefore the conventional Planck-force expression collapses exactly to the QLM lattice form.

Dimensional check. The conventional expression has dimensions

$$\begin{aligned} \left[\frac{c^4}{G}\right] &= \frac{(\mathrm{m\,s^{-1}})^4}{\mathrm{m^3\,kg^{-1}\,s^{-2}}} \\ &= \frac{\mathrm{m^4\,s^{-4}}}{\mathrm{m^3\,kg^{-1}\,s^{-2}}} \\ &= \mathrm{kg\,m\,s^{-2}} = \mathrm{N}. \end{aligned} \tag{4.123}$$

The QLM form has the same dimension,

$$\begin{aligned} \left[\frac{\hbar}{t_\mathrm{P}\ell_\mathrm{P}}\right] &= \frac{\mathrm{kg\,m^2\,s^{-1}}}{(\mathrm{s})(\mathrm{m})} \\ &= \mathrm{kg\,m\,s^{-2}} = \mathrm{N}. \end{aligned} \tag{4.124}$$

The momentum-throughput form also has the same dimension,

$$\left[\frac{p_\mathrm{P}}{t_\mathrm{P}}\right] = \frac{\mathrm{kg\,m\,s^{-1}}}{\mathrm{s}} = \mathrm{kg\,m\,s^{-2}}. \tag{4.125}$$

Numerical evaluation. Using CODATA 2022 values,

$$\begin{aligned} F_{\mathrm{P}} &= \frac{c^4}{G} \\ &= \frac{(2.997\,925 \times 10^{8}\,\mathrm{m\,s^{-1}})^4}{6.674\,300 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}} \\ &= 1.210\,256 \times 10^{44}\,\mathrm{N}. \end{aligned} \tag{4.126}$$

The QLM lattice form gives the same value:

$$\begin{aligned} F_{\mathrm{P}} &= \frac{E_{\mathrm{P}}}{\ell_{\mathrm{P}}} \\ &= \frac{1.956\,081 \times 10^{9}\,\mathrm{J}}{1.616\,255 \times 10^{-35}\,\mathrm{m}} \\ &= 1.210\,256 \times 10^{44}\,\mathrm{N}. \end{aligned} \tag{4.127}$$

Equivalently,

$$\begin{aligned} F_{\mathrm{P}} &= \frac{\hbar}{t_{\mathrm{P}}\ell_{\mathrm{P}}} \\ &= \frac{1.054\,572 \times 10^{-34}\,\mathrm{J\,s}}{(5.391\,247 \times 10^{-44}\,\mathrm{s})(1.616\,255 \times 10^{-35}\,\mathrm{m})} \\ &= 1.210\,256 \times 10^{44}\,\mathrm{N}. \end{aligned} \tag{4.128}$$

Summary. The Planck force identities are

$$F_{\mathrm{P}} = \frac{c^4}{G} = \frac{E_{\mathrm{P}}}{\ell_{\mathrm{P}}} = \frac{p_{\mathrm{P}}}{t_{\mathrm{P}}} = \frac{\hbar}{t_{\mathrm{P}}\ell_{\mathrm{P}}} = \frac{\hbar}{c t_{\mathrm{P}}^2}. \tag{4.129}$$

All forms are algebraically identical and represent the same per-radian Planck force scale in the Quantum Lattice Model.

4.9 Planck Acceleration

The Planck acceleration a_{P} is the characteristic lattice acceleration scale obtained by comparing the invariant lattice speed c to one Planck proper-time tick t_{P}. Using

$$c = \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}}, \tag{4.130}$$

the QLM acceleration scale is

$$a_{\mathrm{P}} = \frac{c}{t_{\mathrm{P}}} = \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}^2}. \tag{4.131}$$

This quantity is a derived Planck-scale rate of velocity change. The causal transport bound remains the lattice speed $c = \ell_{\mathrm{P}}/t_{\mathrm{P}}$.

In conventional Planck units, the same acceleration scale is written as

$$a_{\mathrm{P}} = \sqrt{\frac{c^7}{\hbar G}}. \tag{4.132}$$

The purpose of the following reduction is to show that this square-root form collapses exactly to Eq. (4.131).

Collapse of the conventional square-root form. Use the Planck-length identity

$$\ell_P^2 = \frac{\hbar G}{c^3}. \tag{4.133}$$

Solving Eq. (4.133) for $\hbar G$ gives

$$\hbar G = \ell_P^2 c^3. \tag{4.134}$$

Substituting Eq. (4.134) into Eq. (4.132) gives

$$\begin{aligned} a_P &= \sqrt{\frac{c^7}{\hbar G}} \\ &= \sqrt{\frac{c^7}{\ell_P^2 c^3}} \\ &= \sqrt{\frac{c^4}{\ell_P^2}} \\ &= \frac{c^2}{\ell_P}. \end{aligned} \tag{4.135}$$

Using the lattice transport identity $c = \ell_P / t_P$, this becomes

$$\begin{aligned} a_P &= \frac{(\ell_P/t_P)^2}{\ell_P} \\ &= \frac{\ell_P}{t_P^2} \\ &= \frac{c}{t_P}. \end{aligned} \tag{4.136}$$

Therefore the conventional Planck-acceleration expression collapses exactly to the QLM lattice form.

Dimensional check. The conventional square-root expression has dimensions

$$\begin{aligned} \left[\sqrt{\frac{c^7}{\hbar G}}\right] &= \sqrt{\frac{(\mathrm{m\,s^{-1}})^7}{(\mathrm{kg\,m^2\,s^{-1}})(\mathrm{m^3\,kg^{-1}\,s^{-2}})}} \\ &= \sqrt{\frac{\mathrm{m^7\,s^{-7}}}{\mathrm{m^5\,s^{-3}}}} \\ &= \sqrt{\mathrm{m^2\,s^{-4}}} \\ &= \mathrm{m\,s^{-2}}. \end{aligned} \tag{4.137}$$

The QLM form has the same dimension,

$$\left[\frac{\ell_P}{t_P^2}\right] = \frac{\mathrm{m}}{\mathrm{s^2}} = \mathrm{m\,s^{-2}}. \tag{4.138}$$

The velocity-change form also has the same dimension,

$$\left[\frac{c}{t_P}\right] = \frac{\mathrm{m\,s^{-1}}}{\mathrm{s}} = \mathrm{m\,s^{-2}}. \tag{4.139}$$

Numerical evaluation. Using CODATA 2022 values,

$$a_{\mathrm{P}} = \sqrt{\frac{(2.997\,925 \times 10^{8}\,\mathrm{m\,s^{-1}})^{7}}{(1.054\,572 \times 10^{-34}\,\mathrm{J\,s})(6.674\,300 \times 10^{-11}\,\mathrm{m^{3}\,kg^{-1}\,s^{-2}})}}$$
$$= 5.560\,726 \times 10^{51}\,\mathrm{m\,s^{-2}}. \tag{4.140}$$

The QLM lattice form gives the same value:

$$\begin{aligned} a_{\mathrm{P}} &= \frac{c}{t_{\mathrm{P}}} \\ &= \frac{2.997\,925 \times 10^{8}\,\mathrm{m\,s^{-1}}}{5.391\,247 \times 10^{-44}\,\mathrm{s}} \\ &= 5.560\,726 \times 10^{51}\,\mathrm{m\,s^{-2}}. \end{aligned} \tag{4.141}$$

Equivalently,

$$\begin{aligned} a_{\mathrm{P}} &= \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}^{2}} \\ &= \frac{1.616\,255 \times 10^{-35}\,\mathrm{m}}{(5.391\,247 \times 10^{-44}\,\mathrm{s})^{2}} \\ &= 5.560\,726 \times 10^{51}\,\mathrm{m\,s^{-2}}. \end{aligned} \tag{4.142}$$

Dynamical consistency check. Using the previously derived Planck force and Planck mass,

$$F_{\mathrm{P}} = \frac{\hbar}{t_{\mathrm{P}}\ell_{\mathrm{P}}}, \qquad m_{\mathrm{P}} = \frac{\hbar t_{\mathrm{P}}}{\ell_{\mathrm{P}}^{2}}, \tag{4.143}$$

their ratio gives

$$\begin{aligned} \frac{F_{\mathrm{P}}}{m_{\mathrm{P}}} &= \frac{\hbar/(t_{\mathrm{P}}\ell_{\mathrm{P}})}{\hbar t_{\mathrm{P}}/\ell_{\mathrm{P}}^{2}} \\ &= \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}^{2}} \\ &= a_{\mathrm{P}}. \end{aligned} \tag{4.144}$$

Thus the derived Planck force, mass, and acceleration scales satisfy the ordinary relation $a = F/m$ at the Planck scale.

Summary. The Planck acceleration identities are

$$a_{\mathrm{P}} = \sqrt{\frac{c^{7}}{\hbar G}} = \frac{c^{2}}{\ell_{\mathrm{P}}} = \frac{c}{t_{\mathrm{P}}} = \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}^{2}} = \frac{F_{\mathrm{P}}}{m_{\mathrm{P}}}. \tag{4.145}$$

All forms are algebraically identical and represent the same Planck acceleration scale in the Quantum Lattice Model.

4.10 Planck Power

In the Quantum Lattice Model, the Planck power P_P is the rate at which the saturated Planck energy is transported per lattice tick. Since one saturated tick carries one quantum of reduced action $\hbar$ over one Planck proper-time interval t_P,

$$E_\mathrm{P} = \frac{\hbar}{t_\mathrm{P}}. \tag{4.146}$$

The corresponding power scale is therefore

$$P_\mathrm{P} = \frac{E_\mathrm{P}}{t_\mathrm{P}} = \frac{\hbar}{t_\mathrm{P}^2}. \tag{4.147}$$

This is the QLM lattice form of the Planck power.

In conventional Planck units, the same power scale is written as

$$P_\mathrm{P} = \frac{c^5}{G}. \tag{4.148}$$

The purpose of the following reduction is to show that this expression collapses exactly to the QLM lattice form.

Collapse of the conventional form. Use the Planck-length identity

$$\ell_\mathrm{P}^2 = \frac{\hbar G}{c^3}. \tag{4.149}$$

Solving Eq. (4.149) for G gives

$$G = \frac{\ell_\mathrm{P}^2 c^3}{\hbar}. \tag{4.150}$$

Substituting Eq. (4.150) into Eq. (4.148) gives

$$\begin{aligned} P_\mathrm{P} &= \frac{c^5}{\ell_\mathrm{P}^2 c^3/\hbar} \\ &= \frac{\hbar c^2}{\ell_\mathrm{P}^2}. \end{aligned} \tag{4.151}$$

Using the lattice transport identity $c = \ell_\mathrm{P}/t_\mathrm{P}$, this becomes

$$\begin{aligned} P_\mathrm{P} &= \frac{\hbar}{\ell_\mathrm{P}^2}\left(\frac{\ell_\mathrm{P}}{t_\mathrm{P}}\right)^2 \\ &= \frac{\hbar}{t_\mathrm{P}^2}, \end{aligned} \tag{4.152}$$

matching the QLM lattice form.

Relation to Planck force. Using the QLM Planck force

$$F_\mathrm{P} = \frac{\hbar}{t_\mathrm{P}\ell_\mathrm{P}}, \tag{4.153}$$

and the lattice transport identity $c = \ell_\mathrm{P}/t_\mathrm{P}$, the force–power relation gives

$$F_\mathrm{P} c = \frac{\hbar}{t_\mathrm{P}\ell_\mathrm{P}}\left(\frac{\ell_\mathrm{P}}{t_\mathrm{P}}\right) = \frac{\hbar}{t_\mathrm{P}^2} = P_\mathrm{P}. \tag{4.154}$$

Thus the Planck power is also the Planck force transported at the invariant lattice speed.

Dimensional check. The conventional expression has dimensions

$$\left[\frac{c^5}{G}\right] = \frac{(\mathrm{m\,s^{-1}})^5}{\mathrm{m^3\,kg^{-1}\,s^{-2}}} = \frac{\mathrm{m^5\,s^{-5}}}{\mathrm{m^3\,kg^{-1}\,s^{-2}}} = \mathrm{kg\,m^2\,s^{-3}} = \mathrm{W}. \tag{4.155}$$

The QLM form has the same dimension,

$$\left[\frac{\hbar}{t_\mathrm{P}^2}\right] = \frac{\mathrm{kg\,m^2\,s^{-1}}}{\mathrm{s^2}} = \mathrm{kg\,m^2\,s^{-3}} = \mathrm{W}. \tag{4.156}$$

The force–speed form also has the same dimension,

$$[F_\mathrm{P} c] = (\mathrm{kg\,m\,s^{-2}})(\mathrm{m\,s^{-1}}) = \mathrm{kg\,m^2\,s^{-3}}. \tag{4.157}$$

Numerical evaluation. Using CODATA 2022 values,

$$P_\mathrm{P} = \frac{c^5}{G} = \frac{(2.997\,925\times10^{8}\,\mathrm{m\,s^{-1}})^5}{6.674\,300\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}} = 3.628\,255\times10^{52}\,\mathrm{W}. \tag{4.158}$$

The QLM lattice form gives the same value:

$$P_\mathrm{P} = \frac{\hbar}{t_\mathrm{P}^2} = \frac{1.054\,572\times10^{-34}\,\mathrm{J\,s}}{(5.391\,247\times10^{-44}\,\mathrm{s})^2} = 3.628\,255\times10^{52}\,\mathrm{W}. \tag{4.159}$$

Equivalently,

$$P_\mathrm{P} = \frac{E_\mathrm{P}}{t_\mathrm{P}} = \frac{1.956\,081\times10^{9}\,\mathrm{J}}{5.391\,247\times10^{-44}\,\mathrm{s}} = 3.628\,255\times10^{52}\,\mathrm{W}. \tag{4.160}$$

Summary. The Planck power identities are

$$P_{\rm P} = \frac{c^5}{G} = \frac{E_{\rm P}}{t_{\rm P}} = F_{\rm P} c = \frac{\hbar}{t_{\rm P}^2}. \tag{4.161}$$

All forms are algebraically identical and represent the same per-radian Planck power scale in the Quantum Lattice Model.

4.11 Planck Temperature

In the Quantum Lattice Model, the Planck temperature $T_{\rm P}$ is the thermal conversion of the saturated Planck energy scale. The underlying QLM quantity is the one-tick reduced-action throughput

$$E_{\rm P} = \frac{\hbar}{t_{\rm P}}. \tag{4.162}$$

Using the thermodynamic conversion relation $E = k_B T$ [2, 13], the corresponding temperature scale is

$$T_{\rm P} = \frac{E_{\rm P}}{k_B} = \frac{\hbar}{t_{\rm P} k_B}. \tag{4.163}$$

Here k_B is a thermal unit-conversion constant relating energy to temperature; it is not an additional QLM lattice primitive.

In conventional reduced-action Planck units, the same temperature scale is written as

$$T_{\rm P} = \sqrt{\frac{\hbar c^5}{G k_B^2}}. \tag{4.164}$$

The purpose of the following reduction is to show that this square-root form collapses exactly to the QLM thermal form $T_{\rm P} = \hbar/(t_{\rm P} k_B)$.

Collapse of the conventional square-root form. Use the Planck-length identity

$$\ell_{\rm P}^2 = \frac{\hbar G}{c^3}. \tag{4.165}$$

Solving Eq. (4.165) for G gives

$$G = \frac{\ell_{\rm P}^2 c^3}{\hbar}. \tag{4.166}$$

Substituting Eq. (4.166) into Eq. (4.164) gives

$$\begin{aligned} T_{\rm P} &= \sqrt{\frac{\hbar c^5}{(\ell_{\rm P}^2 c^3/\hbar)\, k_B^2}} \\ &= \sqrt{\frac{\hbar^2 c^2}{\ell_{\rm P}^2 k_B^2}} \\ &= \frac{\hbar c}{\ell_{\rm P} k_B}. \end{aligned} \tag{4.167}$$

Using the lattice transport identity $c = \ell_{\rm P}/t_{\rm P}$, this becomes

$$\begin{aligned} T_{\rm P} &= \frac{\hbar}{\ell_{\rm P} k_B} \left(\frac{\ell_{\rm P}}{t_{\rm P}}\right) \\ &= \frac{\hbar}{t_{\rm P} k_B}. \end{aligned} \tag{4.168}$$

Therefore the conventional Planck-temperature expression collapses exactly to the QLM energy-per-tick expression divided by k_B.

Dimensional check. The conventional square-root expression has dimensions

$$\left[\sqrt{\frac{\hbar c^5}{G k_B^2}}\right] = \sqrt{\frac{(\mathrm{kg\,m^2\,s^{-1}})(\mathrm{m\,s^{-1}})^5}{(\mathrm{m^3\,kg^{-1}\,s^{-2}})(\mathrm{J\,K^{-1}})^2}}$$
$$= \sqrt{\frac{\mathrm{kg\,m^7\,s^{-6}}}{(\mathrm{m^3\,kg^{-1}\,s^{-2}})(\mathrm{kg^2\,m^4\,s^{-4}\,K^{-2}})}}$$
$$= \sqrt{\frac{\mathrm{kg\,m^7\,s^{-6}}}{\mathrm{kg\,m^7\,s^{-6}\,K^{-2}}}}$$
$$= \sqrt{\mathrm{K}^2} = \mathrm{K}. \tag{4.169}$$

The QLM thermal form has the same dimension,

$$\left[\frac{\hbar}{t_\mathrm{P} k_B}\right] = \frac{\mathrm{J\,s}}{(\mathrm{s})(\mathrm{J\,K^{-1}})}$$
$$= \mathrm{K}. \tag{4.170}$$

Numerical evaluation. Using CODATA 2022 values,

$$T_\mathrm{P} = \sqrt{\frac{(1.054\,572\times10^{-34}\,\mathrm{J\,s})(2.997\,925\times10^{8}\,\mathrm{m\,s^{-1}})^5}{(6.674\,300\times10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}})(1.380\,649\times10^{-23}\,\mathrm{J\,K^{-1}})^2}}$$
$$= 1.416\,784\times10^{32}\,\mathrm{K}. \tag{4.171}$$

The QLM thermal form gives the same value:

$$T_\mathrm{P} = \frac{\hbar}{t_\mathrm{P} k_B}$$
$$= \frac{1.054\,572\times10^{-34}\,\mathrm{J\,s}}{(5.391\,247\times10^{-44}\,\mathrm{s})(1.380\,649\times10^{-23}\,\mathrm{J\,K^{-1}})}$$
$$= 1.416\,784\times10^{32}\,\mathrm{K}. \tag{4.172}$$

Equivalently,

$$T_\mathrm{P} = \frac{E_\mathrm{P}}{k_B}$$
$$= \frac{1.956\,081\times10^{9}\,\mathrm{J}}{1.380\,649\times10^{-23}\,\mathrm{J\,K^{-1}}}$$
$$= 1.416\,784\times10^{32}\,\mathrm{K}. \tag{4.173}$$

Dimensionless thermal fraction. A temperature increment ΔT corresponds to a thermal energy increment $k_B \Delta T$. Relative to the Planck energy scale, the dimensionless thermal fraction is

$$\chi_T = \frac{k_B \Delta T}{E_\mathrm{P}} = \Delta T\left(\frac{t_\mathrm{P} k_B}{\hbar}\right). \tag{4.174}$$

For a temperature increment of 1 K,

$$\chi_{1\mathrm{K}} = \frac{k_B}{E_\mathrm{P}} = \frac{1}{T_\mathrm{P}} = \frac{t_\mathrm{P} k_B}{\hbar}. \tag{4.175}$$

Numerically,

$$\chi_{1\mathrm{K}} = \frac{1}{T_{\mathrm{P}}} = 7.058\,239 \times 10^{-33}\,\mathrm{K}^{-1}. \tag{4.176}$$

This quantity measures the fraction of the Planck energy scale represented by one kelvin. It should not be interpreted as coherent phase advance unless a specific thermal phase-transport channel is defined.

Summary. The Planck temperature identities are

$$T_{\mathrm{P}} = \sqrt{\frac{\hbar c^5}{G k_B^2}} = \frac{\hbar c}{\ell_{\mathrm{P}} k_B} = \frac{E_{\mathrm{P}}}{k_B} = \frac{\hbar}{t_{\mathrm{P}} k_B}. \tag{4.177}$$

The one-kelvin dimensionless thermal fraction is

$$\chi_{1\mathrm{K}} = \frac{1}{T_{\mathrm{P}}} = \frac{t_{\mathrm{P}} k_B}{\hbar}. \tag{4.178}$$

All forms are algebraically identical once temperature is treated as the thermal conversion of the QLM Planck energy scale.

5 Unified Planck Lattice Closure

The preceding sections show that the basic Planck mechanical, wave, transport, and thermal scales reduce algebraically to the QLM primitive triplet

$$\{\hbar, \ell_{\mathrm{P}}, t_{\mathrm{P}}\}, \tag{5.1}$$

together with the two structural relations

$$E = \hbar \frac{d\theta}{d\tau}, \tag{5.2}$$

and

$$c = \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}}. \tag{5.3}$$

Equation (5.2) defines energy as reduced-action flow per unit proper time, while Eq. (5.3) defines the invariant lattice transport speed relating one spatial lattice link to one temporal lattice tick.

Once these primitives are specified, the Planck system closes as a minimal algebra. The core wave-sector identities are

$$\omega_P = \frac{1}{t_{\mathrm{P}}}, \qquad k_P = \frac{1}{\ell_{\mathrm{P}}}, \qquad \frac{\omega_P}{k_P} = c. \tag{5.4}$$

The corresponding energy–momentum identities are

$$E_{\mathrm{P}} = \frac{\hbar}{t_{\mathrm{P}}}, \qquad p_{\mathrm{P}} = \frac{\hbar}{\ell_{\mathrm{P}}}, \qquad \frac{E_{\mathrm{P}}}{p_{\mathrm{P}}} = c. \tag{5.5}$$

The inertial and mechanical scales are

$$m_{\mathrm{P}} = \frac{\hbar t_{\mathrm{P}}}{\ell_{\mathrm{P}}^2}, \qquad F_{\mathrm{P}} = \frac{\hbar}{t_{\mathrm{P}} \ell_{\mathrm{P}}}, \qquad a_{\mathrm{P}} = \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}^2}, \qquad P_{\mathrm{P}} = \frac{\hbar}{t_{\mathrm{P}}^2}. \tag{5.6}$$

These satisfy the expected internal consistency relations

$$m_{\mathrm{P}}c^2 = E_{\mathrm{P}}, \qquad F_{\mathrm{P}} = \frac{p_{\mathrm{P}}}{t_{\mathrm{P}}}, \qquad a_{\mathrm{P}} = \frac{F_{\mathrm{P}}}{m_{\mathrm{P}}}, \qquad P_{\mathrm{P}} = F_{\mathrm{P}}c = \frac{E_{\mathrm{P}}}{t_{\mathrm{P}}}. \tag{5.7}$$

The thermal scale is obtained by converting the QLM Planck energy through the Boltzmann relation $E = k_B T$:

$$T_{\mathrm{P}} = \frac{E_{\mathrm{P}}}{k_B} = \frac{\hbar}{t_{\mathrm{P}}k_B}. \tag{5.8}$$

Here k_B is a thermal conversion constant, not an additional QLM lattice primitive.

The Planck energy density, used in the density-cap sector of the QLM, is

$$u_{\mathrm{P}} = \frac{E_{\mathrm{P}}}{\ell_{\mathrm{P}}^3} = \frac{\hbar}{\ell_{\mathrm{P}}^3 t_{\mathrm{P}}}. \tag{5.9}$$

The corresponding mass density is

$$\rho_{\mathrm{P}} = \frac{m_{\mathrm{P}}}{\ell_{\mathrm{P}}^3} = \frac{\hbar t_{\mathrm{P}}}{\ell_{\mathrm{P}}^5}. \tag{5.10}$$

Thus the core Planck lattice closure is

$$\begin{aligned}
E_{\mathrm{P}} &= \frac{\hbar}{t_{\mathrm{P}}}, & p_{\mathrm{P}} &= \frac{\hbar}{\ell_{\mathrm{P}}}, & m_{\mathrm{P}} &= \frac{\hbar t_{\mathrm{P}}}{\ell_{\mathrm{P}}^2}, \\
F_{\mathrm{P}} &= \frac{\hbar}{t_{\mathrm{P}}\ell_{\mathrm{P}}}, & a_{\mathrm{P}} &= \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}^2}, & P_{\mathrm{P}} &= \frac{\hbar}{t_{\mathrm{P}}^2}, \\
T_{\mathrm{P}} &= \frac{\hbar}{t_{\mathrm{P}}k_B}, & \omega_P &= \frac{1}{t_{\mathrm{P}}}, & k_P &= \frac{1}{\ell_{\mathrm{P}}}, \\
u_{\mathrm{P}} &= \frac{\hbar}{\ell_{\mathrm{P}}^3 t_{\mathrm{P}}}, & \rho_{\mathrm{P}} &= \frac{\hbar t_{\mathrm{P}}}{\ell_{\mathrm{P}}^5}.
\end{aligned} \tag{5.11}$$

In this formulation, the conventional square-root combinations of $\{\hbar, G, c\}$ are recovered as derived identities. The underlying QLM structure is instead generated by reduced action $\hbar$, spatial lattice spacing ℓ_{P}, temporal lattice spacing t_{P}, and the phase–transport relations

$$E = \hbar\frac{d\theta}{d\tau}, \qquad c = \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}}.$$

This reduced-action closure forms the Planck-scale reference layer used by the later QLM developments in density saturation, Lorentz kinematics, routing admittance, and quantum phase transport.

Dimensional Closure of the QLM System

The dimensional consistency of the QLM Planck system follows from the primitive assignments

$$[\hbar] = \mathrm{kg\,m^2\,s^{-1}}, \qquad [\ell_{\mathrm{P}}] = \mathrm{m}, \qquad [t_{\mathrm{P}}] = \mathrm{s}. \tag{5.12}$$

The wave-sector dimensions are

$$\left[\frac{1}{t_{\mathrm{P}}}\right] = \mathrm{s^{-1}}, \qquad \left[\frac{1}{\ell_{\mathrm{P}}}\right] = \mathrm{m^{-1}}, \qquad \left[\frac{\omega_P}{k_P}\right] = \mathrm{m\,s^{-1}}. \tag{5.13}$$

The energy–momentum and inertial dimensions are

$$\left[\frac{\hbar}{t_{\rm P}}\right] = {\rm J}, \qquad \left[\frac{\hbar}{\ell_{\rm P}}\right] = {\rm kg\,m\,s^{-1}}, \qquad \left[\frac{\hbar t_{\rm P}}{\ell_{\rm P}^2}\right] = {\rm kg}. \tag{5.14}$$

The dynamical dimensions are

$$\left[\frac{\hbar}{t_{\rm P}\ell_{\rm P}}\right] = {\rm N}, \qquad \left[\frac{\ell_{\rm P}}{t_{\rm P}^2}\right] = {\rm m\,s^{-2}}, \qquad \left[\frac{\hbar}{t_{\rm P}^2}\right] = {\rm W}. \tag{5.15}$$

The thermal and density dimensions are

$$\left[\frac{\hbar}{t_{\rm P}k_B}\right] = {\rm K}, \qquad \left[\frac{\hbar}{\ell_{\rm P}^3 t_{\rm P}}\right] = {\rm J\,m^{-3}}, \qquad \left[\frac{\hbar t_{\rm P}}{\ell_{\rm P}^5}\right] = {\rm kg\,m^{-3}}. \tag{5.16}$$

Electromagnetic quantities close consistently once the per-radian Planck impedance normalization is introduced,

$$Z_{\rm P} = \frac{Z_0}{4\pi}, \qquad Y_{\rm P} = \frac{1}{Z_{\rm P}}. \tag{5.17}$$

The corresponding dimensional checks are

$$\left[\frac{\hbar}{Z_{\rm P}}\right] = {\rm C}^2, \qquad \left[\frac{E_{\rm P}}{q_{\rm P}}\right] = {\rm V}, \qquad \left[\frac{q_{\rm P}}{t_{\rm P}}\right] = {\rm A}. \tag{5.18}$$

Thus the mechanical, thermal, density, and electromagnetic sectors are dimensionally compatible with the primitive QLM lattice set. The constants k_B and $Z_{\rm P}$ function as sector conversion constants; they do not replace the primitive ontology $\{\hbar, \ell_{\rm P}, t_{\rm P}\}$.

6 Geometric Foundations and Planck Densities of the QLM

6.1 Primitive Geometric Cells of the Lattice

In the Quantum Lattice Model, the Planck length $\ell_{\rm P}$ and Planck time $t_{\rm P}$ are the primitive spatial and temporal increments of the discrete lattice. They define the fundamental geometric cells

$$A_P = \ell_{\rm P}^2, \qquad \mathcal{V}_P = \ell_{\rm P}^3, \qquad V_4 = \ell_{\rm P}^3 t_{\rm P}. \tag{6.1}$$

Here A_P is the Planck area, corresponding to the face of a lattice cell; $\mathcal{V}_P$ is the Planck spatial volume; and V_4 is the Planck four-volume associated with one spatial cell evolved through one Planck proper-time tick.

These geometric cells provide the natural reference scales for the density relations derived below. They are not independent primitives; they are powers and products of the primitive lattice increments $\ell_{\rm P}$ and $t_{\rm P}$.

Temporal–spatial locking. The primitive increments are linked by the invariant lattice transport identity

$$c = \frac{\ell_{\rm P}}{t_{\rm P}}. \tag{6.2}$$

The associated angular phase rates are

$$\omega_P = \frac{1}{t_{\rm P}}, \qquad k_P = \frac{1}{\ell_{\rm P}}, \tag{6.3}$$

and therefore satisfy

$$\frac{\omega_P}{k_P} = \frac{1/t_\mathrm{P}}{1/\ell_\mathrm{P}} = \frac{\ell_\mathrm{P}}{t_\mathrm{P}} = c. \tag{6.4}$$

This restates the QLM null phase-transport relation in geometric form.

One radian per four-volume cell. Energy is defined as reduced-action phase flow per unit proper time,

$$E = \hbar \frac{d\theta}{d\tau}. \tag{6.5}$$

For one saturated lattice tick,

$$\Delta\theta = 1, \qquad \Delta\tau = t_\mathrm{P}, \qquad \Delta S = \hbar, \tag{6.6}$$

so the corresponding Planck energy is

$$E_\mathrm{P} = \frac{\Delta S}{\Delta\tau} = \frac{\hbar}{t_\mathrm{P}}. \tag{6.7}$$

Since the four-volume of one spatial lattice cell evolved through one tick is

$$V_4 = \ell_\mathrm{P}^3 t_\mathrm{P}, \tag{6.8}$$

over one Planck proper-time interval, the saturated four-volume cell supports one radian of coherent phase advance and one quantum of reduced action $\hbar$.

6.2 Planck Phase-Density Scale

Over one saturated Planck four-volume cell,

$$V_4 = \ell_\mathrm{P}^3 t_\mathrm{P}, \tag{6.9}$$

the QLM lattice supports one radian of coherent phase advance. The corresponding Planck phase-density scale is therefore

$$\Phi_P = \frac{1}{V_4} = \frac{1}{\ell_\mathrm{P}^3 t_\mathrm{P}}. \tag{6.10}$$

This quantity measures radian-normalized phase advance per unit spatial volume per unit proper time. Since radians are dimensionless, Φ_P has dimensions $\mathrm{m}^{-3}\,\mathrm{s}^{-1}$.

Using the Planck angular frequency

$$\omega_P = \frac{1}{t_\mathrm{P}}, \tag{6.11}$$

the same phase-density scale may be written

$$\Phi_P = \frac{\omega_P}{\ell_\mathrm{P}^3}. \tag{6.12}$$

Using the lattice identity $\ell_\mathrm{P} = ct_\mathrm{P}$, it may also be written as

$$\begin{aligned} \Phi_P &= \frac{1}{\ell_\mathrm{P}^3 t_\mathrm{P}} \\ &= \frac{1}{(ct_\mathrm{P})^3 t_\mathrm{P}} \\ &= \frac{1}{c^3 t_\mathrm{P}^4}. \end{aligned} \tag{6.13}$$

These are equivalent representations of the same derived lattice phase-density scale. The primitive quantities remain $\{\hbar, \ell_\mathrm{P}, t_\mathrm{P}\}$.

Dimensional check. The four-volume form has dimensions

$$[V_4] = [\ell_{\rm P}^3 t_{\rm P}] = \mathrm{m}^3\,\mathrm{s}. \tag{6.14}$$

Therefore

$$[\Phi_P] = \left[\frac{1}{\ell_{\rm P}^3 t_{\rm P}}\right] = \mathrm{m}^{-3}\,\mathrm{s}^{-1}. \tag{6.15}$$

The angular-frequency form has the same dimension,

$$\left[\frac{\omega_P}{\ell_{\rm P}^3}\right] = \frac{\mathrm{s}^{-1}}{\mathrm{m}^3} = \mathrm{m}^{-3}\,\mathrm{s}^{-1}. \tag{6.16}$$

Numerical evaluation. Using CODATA 2022 values,

$$\begin{aligned}\Phi_P &= \frac{1}{(1.616\,255\times10^{-35}\,\mathrm{m})^3(5.391\,247\times10^{-44}\,\mathrm{s})}\\ &= 4.393\,202\times10^{147}\,\mathrm{m}^{-3}\,\mathrm{s}^{-1}.\end{aligned} \tag{6.17}$$

6.3 Planck Energy Density

The QLM Planck energy density $u_{\rm P}$ is the saturated Planck energy per Planck spatial cell. Since

$$E_{\rm P} = \frac{\hbar}{t_{\rm P}}, \tag{6.18}$$

and the Planck spatial volume is

$$\mathcal{V}_P = \ell_{\rm P}^3, \tag{6.19}$$

the corresponding energy density is

$$\begin{aligned}u_{\rm P} &= \frac{E_{\rm P}}{\mathcal{V}_P}\\ &= \frac{\hbar/t_{\rm P}}{\ell_{\rm P}^3}\\ &= \frac{\hbar}{\ell_{\rm P}^3 t_{\rm P}}.\end{aligned} \tag{6.20}$$

Using the phase-density scale from Eq. (6.10), this can also be written as

$$u_{\rm P} = \hbar\,\Phi_P. \tag{6.21}$$

Thus the Planck energy density is reduced action multiplied by the radian-normalized phase-density scale of the lattice.

Relation to Planck power. Since $P_{\rm P} = \hbar/t_{\rm P}^2$ and $c = \ell_{\rm P}/t_{\rm P}$, the same energy density satisfies the transport identity

$$\begin{aligned}u_{\rm P}\,\ell_{\rm P}^2 c &= \left(\frac{\hbar}{\ell_{\rm P}^3 t_{\rm P}}\right)\ell_{\rm P}^2\left(\frac{\ell_{\rm P}}{t_{\rm P}}\right)\\ &= \frac{\hbar}{t_{\rm P}^2}\\ &= P_{\rm P}.\end{aligned} \tag{6.22}$$

This states that transporting the saturated Planck energy density across one lattice face at the invariant lattice speed reproduces the Planck power scale.

Dimensional check. The QLM form has dimensions

$$\begin{aligned}\left[\frac{\hbar}{\ell_{\rm P}^3 t_{\rm P}}\right] &= \frac{\rm kg\,m^2\,s^{-1}}{(\rm m^3)(s)} \\ &= \rm kg\,m^{-1}\,s^{-2} \\ &= \rm J\,m^{-3}. \end{aligned} \tag{6.23}$$

The phase-density form has the same dimension,

$$[\hbar\Phi_P] = (\rm J\,s)(m^{-3}\,s^{-1}) = J\,m^{-3}. \tag{6.24}$$

The transport form also has the correct dimension,

$$[u_{\rm P}\ell_{\rm P}^2 c] = (\rm J\,m^{-3})(m^2)(m\,s^{-1}) = J\,s^{-1} = W. \tag{6.25}$$

Numerical evaluation. Using CODATA 2022 values,

$$\begin{aligned} u_{\rm P} &= \frac{E_{\rm P}}{\ell_{\rm P}^3} \\ &= \frac{1.956\,081\times10^{9}\,\rm J}{(1.616\,255\times10^{-35}\,\rm m)^3} \\ &= 4.632\,947\times10^{113}\,\rm J\,m^{-3}. \end{aligned} \tag{6.26}$$

Equivalently,

$$\begin{aligned} u_{\rm P} &= \hbar\Phi_P \\ &= (1.054\,572\times10^{-34}\,\rm J\,s)(4.393\,202\times10^{147}\,m^{-3}\,s^{-1}) \\ &= 4.632\,947\times10^{113}\,\rm J\,m^{-3}. \end{aligned} \tag{6.27}$$

6.4 Planck Mass Density

The QLM Planck mass density $\rho_{\rm P}$ is the Planck mass per Planck spatial cell. Since

$$m_{\rm P} = \frac{\hbar t_{\rm P}}{\ell_{\rm P}^2}, \tag{6.28}$$

the corresponding mass density is

$$\begin{aligned} \rho_{\rm P} &= \frac{m_{\rm P}}{\ell_{\rm P}^3} \\ &= \frac{\hbar t_{\rm P}}{\ell_{\rm P}^5}. \end{aligned} \tag{6.29}$$

This is the QLM lattice form of the Planck mass density.

Collapse of the conventional form. The conventional reduced-action Planck mass density is

$$\rho_{\mathrm{P}} = \frac{c^5}{\hbar G^2}. \tag{6.30}$$

Use the Planck-length identity

$$\ell_{\mathrm{P}}^2 = \frac{\hbar G}{c^3}. \tag{6.31}$$

Solving Eq. (6.31) for G gives

$$G = \frac{\ell_{\mathrm{P}}^2 c^3}{\hbar}. \tag{6.32}$$

Substituting Eq. (6.32) into Eq. (6.30) gives

$$\begin{aligned} \rho_{\mathrm{P}} &= \frac{c^5}{\hbar \left(\ell_{\mathrm{P}}^2 c^3/\hbar\right)^2} \\ &= \frac{c^5}{\ell_{\mathrm{P}}^4 c^6/\hbar} \\ &= \frac{\hbar}{\ell_{\mathrm{P}}^4 c}. \end{aligned} \tag{6.33}$$

Using the lattice transport identity $c = \ell_{\mathrm{P}}/t_{\mathrm{P}}$, this becomes

$$\begin{aligned} \rho_{\mathrm{P}} &= \frac{\hbar}{\ell_{\mathrm{P}}^4(\ell_{\mathrm{P}}/t_{\mathrm{P}})} \\ &= \frac{\hbar t_{\mathrm{P}}}{\ell_{\mathrm{P}}^5}, \end{aligned} \tag{6.34}$$

matching the QLM lattice form.

Energy–mass closure. Since $E_{\mathrm{P}} = m_{\mathrm{P}} c^2$, division by the common spatial cell volume ℓ_{P}^3 gives

$$u_{\mathrm{P}} = \rho_{\mathrm{P}} c^2. \tag{6.35}$$

This relation also follows directly from the QLM forms:

$$\begin{aligned} \rho_{\mathrm{P}} c^2 &= \left(\frac{\hbar t_{\mathrm{P}}}{\ell_{\mathrm{P}}^5}\right)\left(\frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}}\right)^2 \\ &= \frac{\hbar}{\ell_{\mathrm{P}}^3 t_{\mathrm{P}}} \\ &= u_{\mathrm{P}}. \end{aligned} \tag{6.36}$$

Dimensional check. The conventional expression has dimensions

$$\begin{aligned} \left[\frac{c^5}{\hbar G^2}\right] &= \frac{(\mathrm{m\,s^{-1}})^5}{(\mathrm{kg\,m^2\,s^{-1}})(\mathrm{m^3\,kg^{-1}\,s^{-2}})^2} \\ &= \frac{\mathrm{m^5\,s^{-5}}}{(\mathrm{kg\,m^2\,s^{-1}})(\mathrm{m^6\,kg^{-2}\,s^{-4}})} \\ &= \frac{\mathrm{m^5\,s^{-5}}}{\mathrm{m^8\,kg^{-1}\,s^{-5}}} \\ &= \mathrm{kg\,m^{-3}}. \end{aligned} \tag{6.37}$$

The QLM form has the same dimension,

$$\left[\frac{\hbar t_{\rm P}}{\ell_{\rm P}^5}\right] = \frac{(\mathrm{kg\,m^2\,s^{-1}})(\mathrm{s})}{\mathrm{m^5}}$$
$$= \mathrm{kg\,m^{-3}}. \tag{6.38}$$

The energy–mass closure has the correct dimension,

$$[\rho_{\rm P} c^2] = (\mathrm{kg\,m^{-3}})(\mathrm{m^2\,s^{-2}}) = \mathrm{kg\,m^{-1}\,s^{-2}} = \mathrm{J\,m^{-3}}. \tag{6.39}$$

Numerical evaluation. Using CODATA 2022 values,

$$\rho_{\rm P} = \frac{m_{\rm P}}{\ell_{\rm P}^3}$$
$$= \frac{2.176\,434 \times 10^{-8}\,\mathrm{kg}}{(1.616\,255 \times 10^{-35}\,\mathrm{m})^3}$$
$$= 5.155\,000 \times 10^{96}\,\mathrm{kg\,m^{-3}}. \tag{6.40}$$

Equivalently,

$$\rho_{\rm P} = \frac{\hbar t_{\rm P}}{\ell_{\rm P}^5}$$
$$= \frac{(1.054\,572 \times 10^{-34}\,\mathrm{J\,s})(5.391\,247 \times 10^{-44}\,\mathrm{s})}{(1.616\,255 \times 10^{-35}\,\mathrm{m})^5}$$
$$= 5.155\,000 \times 10^{96}\,\mathrm{kg\,m^{-3}}. \tag{6.41}$$

6.5 Summary of Planck Densities

The QLM density hierarchy is

$$\Phi_P = \frac{1}{\ell_{\rm P}^3 t_{\rm P}}, \qquad u_{\rm P} = \hbar \Phi_P = \frac{\hbar}{\ell_{\rm P}^3 t_{\rm P}}, \qquad \rho_{\rm P} = \frac{m_{\rm P}}{\ell_{\rm P}^3} = \frac{\hbar t_{\rm P}}{\ell_{\rm P}^5}. \tag{6.42}$$

These quantities descend from the same four-volume rule: one saturated Planck cell supports one radian of coherent phase advance and one quantum of reduced action $\hbar$ over one Planck proper-time interval,

$$V_4 = \ell_{\rm P}^3 t_{\rm P}. \tag{6.43}$$

The energy and mass densities are related by

$$u_{\rm P} = \rho_{\rm P} c^2. \tag{6.44}$$

Thus phase density, energy density, and mass density form a consistent Planck-scale density hierarchy generated by the QLM primitive lattice set $\{\hbar, \ell_{\rm P}, t_{\rm P}\}$.

7 Gravitational Constant in Reduced-Action Lattice Form

Gravity in the QLM is expressed using the same reduced-action lattice basis as the other Planck quantities. The primitive set is

$$\{\hbar, \ell_{\rm P}, t_{\rm P}\}, \tag{7.1}$$

together with the invariant lattice transport identity

$$c = \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}}. \tag{7.2}$$

The Newtonian gravitational constant is not taken as a primitive QLM quantity. It is recovered as the derived lattice expression

$$G = \frac{\ell_{\mathrm{P}}^5}{t_{\mathrm{P}}^3 \hbar}. \tag{7.3}$$

This relation is the gravitational conversion factor required for the standard Planck-time and Planck-length definitions to collapse to the QLM primitive lattice scales.

Equivalent Representations of G

Starting from the conventional reduced-action Planck-time relation

$$t_{\mathrm{P}}^2 = \frac{\hbar G}{c^5}, \tag{7.4}$$

solving for G gives

$$G = \frac{c^5 t_{\mathrm{P}}^2}{\hbar}. \tag{7.5}$$

Using the lattice transport identity $c = \ell_{\mathrm{P}}/t_{\mathrm{P}}$, this becomes

$$\begin{aligned} G &= \frac{(\ell_{\mathrm{P}}/t_{\mathrm{P}})^5 t_{\mathrm{P}}^2}{\hbar} \\ &= \frac{\ell_{\mathrm{P}}^5}{t_{\mathrm{P}}^3 \hbar}. \end{aligned} \tag{7.6}$$

Starting instead from the conventional reduced-action Planck-length relation

$$\ell_{\mathrm{P}}^2 = \frac{\hbar G}{c^3}, \tag{7.7}$$

solving for G gives

$$G = \frac{c^3 \ell_{\mathrm{P}}^2}{\hbar}. \tag{7.8}$$

Again using $c = \ell_{\mathrm{P}}/t_{\mathrm{P}}$, this becomes

$$\begin{aligned} G &= \frac{(\ell_{\mathrm{P}}/t_{\mathrm{P}})^3 \ell_{\mathrm{P}}^2}{\hbar} \\ &= \frac{\ell_{\mathrm{P}}^5}{t_{\mathrm{P}}^3 \hbar}. \end{aligned} \tag{7.9}$$

Thus the temporal and spatial Planck routes both reduce to the same QLM lattice form:

$$G = \frac{c^5 t_{\mathrm{P}}^2}{\hbar} = \frac{c^3 \ell_{\mathrm{P}}^2}{\hbar} = \frac{\ell_{\mathrm{P}}^5}{t_{\mathrm{P}}^3 \hbar}. \tag{7.10}$$

Dimensional check. The QLM lattice form has dimensions

$$\left[\frac{\ell_{\rm P}^5}{t_{\rm P}^3\hbar}\right] = \frac{\rm m^5}{(\rm s^3)(\rm kg\,m^2\,s^{-1})} = \rm m^3\,kg^{-1}\,s^{-2}, \tag{7.11}$$

which is the correct SI dimension of the gravitational constant.

The temporal form gives the same dimension:

$$\left[\frac{c^5 t_{\rm P}^2}{\hbar}\right] = \frac{(\rm m\,s^{-1})^5(\rm s^2)}{\rm kg\,m^2\,s^{-1}} = \rm m^3\,kg^{-1}\,s^{-2}. \tag{7.12}$$

The spatial form also agrees:

$$\left[\frac{c^3 \ell_{\rm P}^2}{\hbar}\right] = \frac{(\rm m\,s^{-1})^3(\rm m^2)}{\rm kg\,m^2\,s^{-1}} = \rm m^3\,kg^{-1}\,s^{-2}. \tag{7.13}$$

Numerical evaluation. Using CODATA 2022 values, the temporal form gives

$$\begin{aligned} G &= \frac{c^5 t_{\rm P}^2}{\hbar} \\ &= \frac{(2.997\,925\times 10^{8}\,\rm m\,s^{-1})^5(5.391\,247\times 10^{-44}\,\rm s)^2}{1.054\,572\times 10^{-34}\,\rm J\,s} \\ &= 6.674\,300\times 10^{-11}\,\rm m^3\,kg^{-1}\,s^{-2}. \end{aligned} \tag{7.14}$$

The pure lattice form gives the same value:

$$\begin{aligned} G &= \frac{\ell_{\rm P}^5}{t_{\rm P}^3\hbar} \\ &= \frac{(1.616\,255\times 10^{-35}\,\rm m)^5}{(5.391\,247\times 10^{-44}\,\rm s)^3(1.054\,572\times 10^{-34}\,\rm J\,s)} \\ &= 6.674\,300\times 10^{-11}\,\rm m^3\,kg^{-1}\,s^{-2}. \end{aligned} \tag{7.15}$$

This confirms numerical agreement with the CODATA 2022 recommended value.

Schwarzschild and Gravitational-Impedance Compatibility

The static exterior gravitational redshift factor of Schwarzschild geometry is

$$Z_g(r) = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2}. \tag{7.16}$$

Within QLM notation this same factor is interpreted as the gravitational impedance factor associated with the exterior Schwarzschild field.

Defining the Schwarzschild radius

$$r_s = \frac{2GM}{c^2}, \tag{7.17}$$

Eq. (7.16) becomes

$$Z_g(r) = \left(1 - \frac{r_s}{r}\right)^{-1/2}. \tag{7.18}$$

Using the QLM lattice representation

$$G = \frac{\ell_{\rm P}^5}{t_{\rm P}^3 \hbar}, \qquad c = \frac{\ell_{\rm P}}{t_{\rm P}}, \tag{7.19}$$

the Schwarzschild factor reduces as follows:

$$\begin{aligned} \frac{2GM}{c^2 r} &= \frac{2M}{r}\left(\frac{\ell_{\rm P}^5}{t_{\rm P}^3 \hbar}\right)\left(\frac{t_{\rm P}^2}{\ell_{\rm P}^2}\right) \\ &= 2\,\frac{M\ell_{\rm P}^3}{\hbar t_{\rm P} r}. \end{aligned} \tag{7.20}$$

Therefore

$$Z_g(r) = \left[1 - 2\,\frac{M\ell_{\rm P}^3}{\hbar t_{\rm P} r}\right]^{-1/2}. \tag{7.21}$$

Using the QLM Planck mass

$$m_{\rm P} = \frac{\hbar t_{\rm P}}{\ell_{\rm P}^2}, \tag{7.22}$$

we have

$$\frac{1}{m_{\rm P}} = \frac{\ell_{\rm P}^2}{\hbar t_{\rm P}}. \tag{7.23}$$

Then

$$2\,\frac{M\ell_{\rm P}^3}{\hbar t_{\rm P} r} = 2\left(\frac{M}{m_{\rm P}}\right)\left(\frac{\ell_{\rm P}}{r}\right). \tag{7.24}$$

Thus the gravitational impedance factor can be written in pure Planck lattice form as

$$Z_g(r) = \left[1 - 2\left(\frac{M}{m_{\rm P}}\right)\left(\frac{\ell_{\rm P}}{r}\right)\right]^{-1/2}. \tag{7.25}$$

The Schwarzschild radius likewise becomes

$$\begin{aligned} r_s &= \frac{2GM}{c^2} \\ &= 2\,\frac{M\ell_{\rm P}^3}{\hbar t_{\rm P}} \\ &= 2\left(\frac{M}{m_{\rm P}}\right)\ell_{\rm P}. \end{aligned} \tag{7.26}$$

Dimensional check. The Schwarzschild radius expression has dimensions

$$\begin{aligned} \left[\frac{GM}{c^2}\right] &= \frac{(\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2})(\mathrm{kg})}{\mathrm{m}^2\,\mathrm{s}^{-2}} \\ &= \mathrm{m}. \end{aligned} \tag{7.27}$$

The QLM lattice form has the same dimension,

$$\left[\frac{M\ell_{\mathrm{P}}^3}{\hbar t_{\mathrm{P}}}\right] = \frac{(\mathrm{kg})(\mathrm{m}^3)}{(\mathrm{kg\,m^2\,s^{-1}})(\mathrm{s})} \\ = \mathrm{m}. \tag{7.28}$$

The dimensionless redshift argument also checks:

$$\left[\left(\frac{M}{m_{\mathrm{P}}}\right)\left(\frac{\ell_{\mathrm{P}}}{r}\right)\right] = 1. \tag{7.29}$$

Equivalence summary. The gravitational redshift factor may therefore be written equivalently as

$$\begin{aligned} Z_g(r) &= \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} \\ &= \left(1 - \frac{r_s}{r}\right)^{-1/2} \\ &= \left[1 - 2\,\frac{M\ell_{\mathrm{P}}^3}{\hbar t_{\mathrm{P}} r}\right]^{-1/2} \\ &= \left[1 - 2\left(\frac{M}{m_{\mathrm{P}}}\right)\left(\frac{\ell_{\mathrm{P}}}{r}\right)\right]^{-1/2}. \end{aligned} \tag{7.30}$$

The QLM gravitational impedance factor is therefore algebraically equivalent to the exterior Schwarzschild redshift factor when expressed in reduced-action Planck lattice units.

8 Charge and Electromagnetic Planck Units in the QLM

In the Quantum Lattice Model, the electromagnetic Planck units are organized around the same reduced-action scale used throughout the mechanical and gravitational sectors. The underlying QLM energy scale is

$$E_{\mathrm{P}} = \frac{\hbar}{t_{\mathrm{P}}}. \tag{8.1}$$

Electromagnetic quantities are obtained by combining this action scale with the vacuum electromagnetic response normalization. The vacuum impedance Z_0 and vacuum permittivity ϵ_0 are SI electromagnetic conversion quantities; they do not replace the QLM primitive lattice set $\{\hbar, \ell_{\mathrm{P}}, t_{\mathrm{P}}\}$.

8.1 Planck Charge and Per-Radian Impedance

In rationalized SI form, the Planck charge is

$$q_{\mathrm{P}}^2 = 4\pi\epsilon_0\hbar c. \tag{8.2}$$

Using the vacuum impedance relation

$$Z_0 = \frac{1}{\epsilon_0 c}, \tag{8.3}$$

Eq. (8.2) becomes

$$q_{\mathrm{P}}^2 = 4\pi\epsilon_0\hbar c$$
$$= \frac{4\pi\hbar}{Z_0}. \tag{8.4}$$

This motivates the per-radian Planck impedance normalization

$$Z_{\mathrm{P}} = \frac{Z_0}{4\pi}, \tag{8.5}$$

so that the charge–action identity becomes

$$q_{\mathrm{P}}^2 = \frac{\hbar}{Z_{\mathrm{P}}} = \hbar Y_{\mathrm{P}}, \qquad Y_{\mathrm{P}} \equiv \frac{1}{Z_{\mathrm{P}}}. \tag{8.6}$$

Equivalently,

$$\hbar = Z_{\mathrm{P}} q_{\mathrm{P}}^2. \tag{8.7}$$

A full-cycle impedance convention may also be defined by aggregating over a 2π phase loop:

$$Z_P^{(\mathrm{loop})} = 2\pi Z_{\mathrm{P}} = \frac{Z_0}{2}, \qquad Z_{\mathrm{P}} = \frac{Z_P^{(\mathrm{loop})}}{2\pi}. \tag{8.8}$$

The per-radian quantity $Z_{\mathrm{P}} = Z_0/(4\pi)$ is the QLM normalization used below.

Dimensional check. The charge identity has dimensions

$$\left[\frac{\hbar}{Z_{\mathrm{P}}}\right] = \frac{\mathrm{J\,s}}{\Omega}$$
$$= \frac{\mathrm{V\,C\,s}}{\mathrm{V\,A^{-1}}}$$
$$= \mathrm{C\,s\,A}$$
$$= \mathrm{C}^2, \tag{8.9}$$

since $1\,\mathrm{A} = 1\,\mathrm{C\,s^{-1}}$. Therefore $q_{\mathrm{P}}^2 = \hbar/Z_{\mathrm{P}}$ has the correct dimension of charge squared.

The impedance normalization has dimensions

$$[Z_{\mathrm{P}}] = [Z_0] = \Omega. \tag{8.10}$$

8.2 The Planck Electromagnetic Quartet

The Planck voltage is energy per unit charge:

$$V_P = \frac{E_{\mathrm{P}}}{q_{\mathrm{P}}} = \frac{\hbar}{t_{\mathrm{P}} q_{\mathrm{P}}}. \tag{8.11}$$

The Planck current is charge transported per Planck tick:

$$I_P = \frac{q_{\mathrm{P}}}{t_{\mathrm{P}}}. \tag{8.12}$$

Their ratio gives the per-radian Planck impedance:

$$\begin{aligned}\frac{V_P}{I_P} &= \frac{E_\mathrm{P}/q_\mathrm{P}}{q_\mathrm{P}/t_\mathrm{P}} \\ &= \frac{E_\mathrm{P} t_\mathrm{P}}{q_\mathrm{P}^2} \\ &= \frac{\hbar}{q_\mathrm{P}^2} \\ &= Z_\mathrm{P}.\end{aligned} \tag{8.13}$$

Thus

$$Z_\mathrm{P} = \frac{V_P}{I_P} = \frac{\hbar}{q_\mathrm{P}^2} = \frac{Z_0}{4\pi}. \tag{8.14}$$

The Planck power is

$$P_\mathrm{P} = \frac{E_\mathrm{P}}{t_\mathrm{P}} = \frac{\hbar}{t_\mathrm{P}^2}. \tag{8.15}$$

The voltage–current product reproduces the same power:

$$\begin{aligned}V_P I_P &= \left(\frac{E_\mathrm{P}}{q_\mathrm{P}}\right)\left(\frac{q_\mathrm{P}}{t_\mathrm{P}}\right) \\ &= \frac{E_\mathrm{P}}{t_\mathrm{P}} \\ &= P_\mathrm{P}.\end{aligned} \tag{8.16}$$

Equivalently,

$$V_P = \sqrt{P_\mathrm{P} Z_\mathrm{P}}, \qquad I_P = \sqrt{\frac{P_\mathrm{P}}{Z_\mathrm{P}}}. \tag{8.17}$$

The per-radian electromagnetic Planck quartet is therefore

$$\{Z_\mathrm{P},\ V_P,\ I_P,\ P_\mathrm{P}\}. \tag{8.18}$$

It closes through the identities

$$Z_\mathrm{P} = \frac{V_P}{I_P}, \qquad P_\mathrm{P} = V_P I_P, \qquad E_\mathrm{P} = V_P q_\mathrm{P}, \qquad I_P = \frac{q_\mathrm{P}}{t_\mathrm{P}}. \tag{8.19}$$

Dimensional check. Voltage has the correct dimension:

$$[V_P] = \left[\frac{E_\mathrm{P}}{q_\mathrm{P}}\right] = \frac{\mathrm{J}}{\mathrm{C}} = \mathrm{V}. \tag{8.20}$$

Current has the correct dimension:

$$[I_P] = \left[\frac{q_\mathrm{P}}{t_\mathrm{P}}\right] = \frac{\mathrm{C}}{\mathrm{s}} = \mathrm{A}. \tag{8.21}$$

The impedance ratio has the correct dimension:

$$\left[\frac{V_P}{I_P}\right] = \frac{\mathrm{V}}{\mathrm{A}} = \Omega. \tag{8.22}$$

The power relation has the correct dimension:

$$[V_P I_P] = (\mathrm{V})(\mathrm{A}) = \mathrm{W}. \tag{8.23}$$

Summary. The electromagnetic Planck-unit identities are

$$q_{\mathrm{P}}^2 = 4\pi\epsilon_0\hbar c = \frac{\hbar}{Z_{\mathrm{P}}}, \qquad Z_{\mathrm{P}} = \frac{Z_0}{4\pi}, \qquad Y_{\mathrm{P}} = \frac{1}{Z_{\mathrm{P}}}. \tag{8.24}$$

The corresponding per-radian voltage, current, and power scales are

$$V_P = \frac{E_{\mathrm{P}}}{q_{\mathrm{P}}}, \qquad I_P = \frac{q_{\mathrm{P}}}{t_{\mathrm{P}}}, \qquad P_{\mathrm{P}} = V_P I_P = \frac{\hbar}{t_{\mathrm{P}}^2}. \tag{8.25}$$

Thus the electromagnetic Planck units close algebraically through the same QLM reduced-action throughput $E_{\mathrm{P}} = \hbar/t_{\mathrm{P}}$, supplemented by the per-radian impedance normalization $Z_{\mathrm{P}} = Z_0/(4\pi)$.

8.3 Fine-Structure Constant and Hydrogenic Calibration

The fine-structure constant α is not a primitive parameter in the QLM lattice ontology. Instead, it appears as a dimensionless calibration factor connecting hydrogenic orbital kinematics, electromagnetic coupling, and the per-radian impedance normalization.

Hydrogenic action identity. The Bohr ground-state action identity is

$$\hbar = m_e v_B a_0, \tag{8.26}$$

where m_e is the electron mass, v_B is the Bohr orbital speed, and a_0 is the Bohr radius. Solving for the Bohr speed gives

$$v_B = \frac{\hbar}{m_e a_0}. \tag{8.27}$$

Using the invariant lattice transport speed

$$c = \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}}, \tag{8.28}$$

the fine-structure constant may be written in hydrogenic kinematic form as

$$\alpha = \frac{v_B}{c}. \tag{8.29}$$

Thus α measures the Bohr orbital speed as a fraction of the invariant lattice propagation speed.

Combining Eqs. (8.26) and (8.29) also gives

$$\hbar = m_e \alpha c a_0. \tag{8.30}$$

This is the hydrogenic calibration form of the reduced-action quantum.

Electromagnetic impedance form. In rationalized SI units, the fine-structure constant is

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}. \tag{8.31}$$

Using the vacuum impedance relation

$$Z_0 = \frac{1}{\epsilon_0 c}, \tag{8.32}$$

and the per-radian Planck impedance normalization

$$Z_{\mathrm{P}} = \frac{Z_0}{4\pi} = \frac{1}{4\pi\epsilon_0 c}, \tag{8.33}$$

we have

$$4\pi\epsilon_0 c = \frac{1}{Z_{\mathrm{P}}}. \tag{8.34}$$

Therefore

$$4\pi\epsilon_0 \hbar c = \frac{\hbar}{Z_{\mathrm{P}}}. \tag{8.35}$$

Substituting Eq. (8.35) into Eq. (8.31) gives

$$\alpha = \frac{e^2 Z_{\mathrm{P}}}{\hbar}. \tag{8.36}$$

Using the Planck charge identity

$$q_{\mathrm{P}}^2 = \frac{\hbar}{Z_{\mathrm{P}}}, \tag{8.37}$$

Eq. (8.36) becomes

$$\alpha = \frac{e^2}{q_{\mathrm{P}}^2}. \tag{8.38}$$

Equivalently,

$$e^2 = \alpha q_{\mathrm{P}}^2, \qquad e = q_{\mathrm{P}}\sqrt{\alpha}. \tag{8.39}$$

Unified calibration identity. The hydrogenic velocity ratio and the electromagnetic impedance ratio are therefore equivalent representations of the same dimensionless constant:

$$\frac{v_B}{c} = \alpha = \frac{e^2 Z_{\mathrm{P}}}{\hbar} = \frac{e^2}{q_{\mathrm{P}}^2}. \tag{8.40}$$

Thus α acts as a calibration bridge between orbital confinement and electromagnetic coupling within the same reduced-action framework. This section does not derive the numerical value of α from first principles; it shows that the standard hydrogenic and electromagnetic forms close consistently in QLM variables.

Dimensional check. The hydrogenic action identity has dimensions

$$\begin{aligned}[m_e v_B a_0] &= (\mathrm{kg})(\mathrm{m\,s^{-1}})(\mathrm{m}) \\ &= \mathrm{kg\,m^2\,s^{-1}} = [\hbar]. \end{aligned} \tag{8.41}$$

The velocity ratio is dimensionless:

$$\left[\frac{v_B}{c}\right] = \frac{\mathrm{m\,s^{-1}}}{\mathrm{m\,s^{-1}}} = 1. \tag{8.42}$$

The impedance form is also dimensionless:

$$\left[\frac{e^2 Z_{\rm P}}{\hbar}\right] = \frac{(\mathrm{C}^2)(\Omega)}{\mathrm{J\,s}} = \frac{(\mathrm{C}^2)(\mathrm{V\,A}^{-1})}{\mathrm{J\,s}} = \frac{(\mathrm{C}^2)(\mathrm{V\,s\,C}^{-1})}{\mathrm{J\,s}} = \frac{\mathrm{C\,V\,s}}{\mathrm{J\,s}} = \frac{\mathrm{J\,s}}{\mathrm{J\,s}} = 1. \tag{8.43}$$

The charge-ratio form is dimensionless as well:

$$\left[\frac{e^2}{q_{\rm P}^2}\right] = \frac{\mathrm{C}^2}{\mathrm{C}^2} = 1. \tag{8.44}$$

Numerical evaluation. Using CODATA 2022 values,

$$\alpha = \frac{v_B}{c} = \frac{2.187\,691 \times 10^6\ \mathrm{m\,s^{-1}}}{2.997\,925 \times 10^8\ \mathrm{m\,s^{-1}}} = 7.297\,353 \times 10^{-3}. \tag{8.45}$$

Equivalently,

$$\alpha^{-1} = 1.370\,360 \times 10^2. \tag{8.46}$$

Summary. The fine-structure calibration identities are

$$\alpha = \frac{v_B}{c} = \frac{e^2 Z_{\rm P}}{\hbar} = \frac{e^2}{q_{\rm P}^2}, \qquad e = q_{\rm P}\sqrt{\alpha}. \tag{8.47}$$

The Bohr identity may therefore be written as

$$\hbar = m_e v_B a_0 = m_e \alpha c a_0. \tag{8.48}$$

These relations show that hydrogenic action, electromagnetic coupling, and per-radian impedance normalization are mutually consistent within the QLM Planck-unit framework.

9 Rydberg Scale in Bohr and Planck–Lattice Form

The Rydberg scale provides a useful hydrogenic calibration of the reduced-action framework. It is not a new QLM primitive. Instead, it shows how the Bohr action identity, the fine-structure velocity ratio, and the lattice transport identity combine consistently.

9.1 Hydrogenic Wavenumber Form

The Bohr ground-state identities are

$$v_B = \alpha c, \qquad \hbar = m_e v_B a_0, \tag{9.1}$$

where v_B is the Bohr speed, a_0 is the Bohr radius, and α is the fine-structure constant.

The standard spectroscopic Rydberg wavenumber is

$$R_\infty = \frac{\alpha}{4\pi a_0}. \tag{9.2}$$

Using the QLM lattice transport identity

$$c = \frac{\ell_P}{t_P}, \tag{9.3}$$

the fine-structure constant may be written as

$$\alpha = \frac{v_B}{c} = \frac{v_B t_P}{\ell_P}. \tag{9.4}$$

Substituting Eq. (9.4) into Eq. (9.2) gives

$$R_\infty = \frac{v_B t_P}{4\pi a_0 \ell_P}. \tag{9.5}$$

This expression is algebraically identical to the standard hydrogenic form. The Planck-lattice quantities enter only through the velocity ratio $v_B/c = v_B t_P/\ell_P$, not as an independent length scale setting R_∞.

The corresponding angular, per-radian wavenumber is

$$k_\infty = 2\pi R_\infty = \frac{\alpha}{2a_0} = \frac{v_B}{2a_0 c}. \tag{9.6}$$

Dimensional check. The Rydberg wavenumber has dimension

$$\left[\frac{\alpha}{4\pi a_0}\right] = \frac{1}{\mathrm{m}} = \mathrm{m}^{-1}, \tag{9.7}$$

since α and 4π are dimensionless. The lattice form has the same dimension:

$$\begin{aligned}\left[\frac{v_B t_P}{4\pi a_0 \ell_P}\right] &= \frac{(\mathrm{m\,s^{-1}})(\mathrm{s})}{(\mathrm{m})(\mathrm{m})} \\ &= \mathrm{m}^{-1}.\end{aligned} \tag{9.8}$$

The angular wavenumber $k_\infty = 2\pi R_\infty$ also has dimension m^{-1}.

Numerical evaluation. Using CODATA 2022 values,

$$\begin{aligned}R_\infty &= \frac{\alpha}{4\pi a_0} \\ &= \frac{7.297\,353 \times 10^{-3}}{4\pi\,(5.291\,772 \times 10^{-11}\,\mathrm{m})} \\ &= 1.097\,373 \times 10^{7}\,\mathrm{m}^{-1}.\end{aligned} \tag{9.9}$$

The corresponding angular wavenumber is

$$\begin{aligned}k_\infty &= 2\pi R_\infty \\ &= 2\pi \left(1.097\,373 \times 10^{7}\,\mathrm{m}^{-1}\right) \\ &= 6.895\,000 \times 10^{7}\,\mathrm{m}^{-1}.\end{aligned} \tag{9.10}$$

9.2 Rydberg Frequency and Energy Form

The Rydberg frequency is defined by

$$\nu_R = cR_\infty. \tag{9.11}$$

Using the standard hydrogenic expression

$$R_\infty = \frac{\alpha^2 m_e c}{2h}, \tag{9.12}$$

gives

$$\nu_R = \frac{\alpha^2 m_e c^2}{2h}. \tag{9.13}$$

Since

$$\alpha = \frac{v_B}{c}, \qquad h = 2\pi\hbar, \tag{9.14}$$

Eq. (9.13) becomes

$$\begin{aligned}\nu_R &= \frac{(v_B^2/c^2)m_e c^2}{2(2\pi\hbar)} \\ &= \frac{m_e v_B^2}{4\pi\hbar}.\end{aligned} \tag{9.15}$$

This form makes the Rydberg frequency a hydrogenic kinetic-energy scale expressed in full-cycle frequency units.

Indeed, with the Bohr kinetic energy

$$E_B = \frac{1}{2} m_e v_B^2, \tag{9.16}$$

Eq. (9.15) becomes

$$\nu_R = \frac{E_B}{2\pi\hbar} = \frac{E_B}{h}. \tag{9.17}$$

Thus the Rydberg frequency is the Bohr kinetic energy expressed in units of the full-cycle action h.

The corresponding angular Rydberg frequency is

$$\omega_R = 2\pi\nu_R = \frac{E_B}{\hbar} = \frac{m_e v_B^2}{2\hbar}. \tag{9.18}$$

This is the per-radian form consistent with the QLM convention $E = \hbar\omega$.

Dimensional check. The Rydberg frequency has dimension

$$[cR_\infty] = (\mathrm{m\,s^{-1}})(\mathrm{m^{-1}}) = \mathrm{s^{-1}}. \tag{9.19}$$

The Bohr kinetic-energy form has the same dimension:

$$\begin{aligned}\left[\frac{m_e v_B^2}{4\pi\hbar}\right] &= \frac{(\mathrm{kg})(\mathrm{m^2\,s^{-2}})}{\mathrm{kg\,m^2\,s^{-1}}} \\ &= \mathrm{s^{-1}}.\end{aligned} \tag{9.20}$$

The angular form also has dimension $\mathrm{s^{-1}}$, with radians dimensionless:

$$\left[\frac{E_B}{\hbar}\right] = \frac{\mathrm{J}}{\mathrm{J\,s}} = \mathrm{s^{-1}}. \tag{9.21}$$

Numerical evaluation. Using CODATA 2022 values,

$$\begin{aligned}\nu_R &= cR_\infty \\ &= \left(2.997\,925\times 10^{8}\,\mathrm{m\,s^{-1}}\right)\left(1.097\,373\times 10^{7}\,\mathrm{m^{-1}}\right) \\ &= 3.289\,842\times 10^{15}\,\mathrm{Hz}.\end{aligned}\tag{9.22}$$

Equivalently, using the Bohr speed form,

$$\begin{aligned}\nu_R &= \frac{m_e v_B^2}{4\pi\hbar} \\ &= \frac{(9.109\,384\times 10^{-31}\,\mathrm{kg})(2.187\,691\times 10^{6}\,\mathrm{m\,s^{-1}})^2}{4\pi(1.054\,572\times 10^{-34}\,\mathrm{J\,s})} \\ &= 3.289\,842\times 10^{15}\,\mathrm{Hz}.\end{aligned}\tag{9.23}$$

Summary. The Rydberg calibration identities are

$$R_\infty = \frac{\alpha}{4\pi a_0} = \frac{v_B t_\mathrm{P}}{4\pi a_0 \ell_\mathrm{P}}, \qquad k_\infty = 2\pi R_\infty = \frac{\alpha}{2a_0}.\tag{9.24}$$

The corresponding frequency identities are

$$\nu_R = cR_\infty = \frac{m_e v_B^2}{4\pi\hbar} = \frac{E_B}{h}, \qquad \omega_R = 2\pi\nu_R = \frac{E_B}{\hbar}.\tag{9.25}$$

These relations show that the Rydberg scale is a hydrogenic calibration of the QLM reduced-action framework, with the Planck lattice appearing through the invariant speed ratio $c = \ell_\mathrm{P}/t_\mathrm{P}$.

9.3 Vacuum Impedance, Admittance, Permittivity, and Permeability

The electromagnetic constants of free space can be expressed in lattice form once the vacuum impedance Z_0 and the invariant lattice speed $c = \ell_\mathrm{P}/t_\mathrm{P}$ are specified. In SI units,

$$Z_0 = \mu_0 c = \frac{1}{\epsilon_0 c}, \qquad c = \frac{\ell_\mathrm{P}}{t_\mathrm{P}}.\tag{9.26}$$

Solving these relations for the vacuum permeability and permittivity gives

$$\mu_0 = \frac{Z_0}{c} = Z_0\frac{t_\mathrm{P}}{\ell_\mathrm{P}},\tag{9.27}$$

$$\epsilon_0 = \frac{1}{Z_0 c} = \frac{t_\mathrm{P}}{Z_0\ell_\mathrm{P}}.\tag{9.28}$$

Thus μ_0 and ϵ_0 are compatible with the QLM lattice transport identity once Z_0 is used as the electromagnetic vacuum response scale.

Maxwell wave-speed consistency. Substituting Eqs. (9.27) and (9.28) gives

$$\begin{aligned}\mu_0\epsilon_0 c^2 &= \left(Z_0\frac{t_\mathrm{P}}{\ell_\mathrm{P}}\right)\left(\frac{t_\mathrm{P}}{Z_0\ell_\mathrm{P}}\right)\left(\frac{\ell_\mathrm{P}}{t_\mathrm{P}}\right)^2 \\ &= 1.\end{aligned}\tag{9.29}$$

Equivalently,

$$\frac{1}{\sqrt{\mu_0\epsilon_0}} = c = \frac{\ell_\mathrm{P}}{t_\mathrm{P}}.\tag{9.30}$$

This shows that the Maxwell vacuum wave-speed relation is consistent with the QLM lattice transport identity.

Impedance and admittance form. Using the per-radian normalization

$$Z_{\mathrm{P}} = \frac{Z_0}{4\pi}, \qquad Y_{\mathrm{P}} = \frac{1}{Z_{\mathrm{P}}}, \tag{9.31}$$

the Planck charge identity is

$$q_{\mathrm{P}}^2 = \frac{\hbar}{Z_{\mathrm{P}}} = \hbar Y_{\mathrm{P}}. \tag{9.32}$$

Equivalently,

$$Z_0 = 4\pi Z_{\mathrm{P}} = \frac{4\pi\hbar}{q_{\mathrm{P}}^2}. \tag{9.33}$$

The vacuum permeability and permittivity may therefore also be written as

$$\mu_0 = \frac{Z_0}{c} = \frac{4\pi Z_{\mathrm{P}}}{c}, \tag{9.34}$$

$$\epsilon_0 = \frac{1}{Z_0 c} = \frac{1}{4\pi Z_{\mathrm{P}} c}. \tag{9.35}$$

Dimensional check. The permeability expression has dimensions

$$\begin{aligned}\left[\frac{Z_0}{c}\right] &= \frac{\Omega}{\mathrm{m\,s^{-1}}} \\ &= \Omega\,\mathrm{s\,m^{-1}} \\ &= \mathrm{H\,m^{-1}},\end{aligned} \tag{9.36}$$

which is the SI dimension of μ_0. The permittivity expression has dimensions

$$\begin{aligned}\left[\frac{1}{Z_0 c}\right] &= \frac{1}{\Omega\,\mathrm{m\,s^{-1}}} \\ &= \frac{\mathrm{s}}{\Omega\,\mathrm{m}} \\ &= \mathrm{F\,m^{-1}},\end{aligned} \tag{9.37}$$

which is the SI dimension of ϵ_0. Finally,

$$[Z_{\mathrm{P}}] = \Omega, \qquad [Y_{\mathrm{P}}] = \Omega^{-1} = \mathrm{S}. \tag{9.38}$$

Summary. The vacuum electromagnetic response relations are

$$Z_0 = \mu_0 c = \frac{1}{\epsilon_0 c}, \qquad \mu_0 = Z_0 \frac{t_{\mathrm{P}}}{\ell_{\mathrm{P}}}, \qquad \epsilon_0 = \frac{t_{\mathrm{P}}}{Z_0 \ell_{\mathrm{P}}}. \tag{9.39}$$

The corresponding per-radian QLM impedance and admittance are

$$Z_{\mathrm{P}} = \frac{Z_0}{4\pi}, \qquad Y_{\mathrm{P}} = \frac{1}{Z_{\mathrm{P}}}, \qquad q_{\mathrm{P}}^2 = \frac{\hbar}{Z_{\mathrm{P}}}. \tag{9.40}$$

Thus the electromagnetic vacuum constants are compatible with the QLM lattice speed $c = \ell_{\mathrm{P}}/t_{\mathrm{P}}$, while charge enters through the reduced-action impedance relation $q_{\mathrm{P}}^2 = \hbar/Z_{\mathrm{P}}$.

9.4 Planck Electromagnetic Field Strengths

The QLM Planck energy density is

$$u_{\mathrm{P}} = \frac{E_{\mathrm{P}}}{\ell_{\mathrm{P}}^3} = \frac{\hbar}{\ell_{\mathrm{P}}^3 t_{\mathrm{P}}}. \tag{9.41}$$

An electromagnetic field-strength scale may be obtained by asking what vacuum electric and magnetic field amplitudes correspond to this energy density.

For a vacuum electromagnetic plane wave, the electric and magnetic contributions are equal, so the total energy density is

$$u_{\mathrm{EM}} = \frac{1}{2}\epsilon_0 \mathcal{E}^2 + \frac{\mathcal{B}^2}{2\mu_0} = \epsilon_0 \mathcal{E}^2 = \frac{\mathcal{B}^2}{\mu_0}, \tag{9.42}$$

with

$$\mathcal{B} = \frac{\mathcal{E}}{c}. \tag{9.43}$$

Setting $u_{\mathrm{EM}} = u_{\mathrm{P}}$ gives the QLM Planck electromagnetic field amplitudes

$$\mathcal{E}_P = \sqrt{\frac{u_{\mathrm{P}}}{\epsilon_0}} = \sqrt{\frac{\hbar}{\epsilon_0 \ell_{\mathrm{P}}^3 t_{\mathrm{P}}}}, \tag{9.44}$$

$$\mathcal{B}_P = \sqrt{\mu_0 u_{\mathrm{P}}} = \sqrt{\frac{\mu_0 \hbar}{\ell_{\mathrm{P}}^3 t_{\mathrm{P}}}}. \tag{9.45}$$

These are derived electromagnetic field scales associated with assigning the Planck energy density to a vacuum plane-wave field configuration. They are not additional QLM primitives.

Wave-speed and impedance relations. The ratio of the field amplitudes is

$$\begin{aligned} \frac{\mathcal{E}_P}{\mathcal{B}_P} &= \frac{\sqrt{u_{\mathrm{P}}/\epsilon_0}}{\sqrt{\mu_0 u_{\mathrm{P}}}} \\ &= \frac{1}{\sqrt{\epsilon_0 \mu_0}} \\ &= c = \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}}. \end{aligned} \tag{9.46}$$

Thus the vacuum field-amplitude ratio is consistent with the QLM lattice transport identity.

The magnetic field intensity is

$$\mathcal{H}_P = \frac{\mathcal{B}_P}{\mu_0}. \tag{9.47}$$

Therefore

$$\begin{aligned} \frac{\mathcal{E}_P}{\mathcal{H}_P} &= \frac{\mathcal{E}_P}{\mathcal{B}_P/\mu_0} \\ &= \mu_0 \frac{\mathcal{E}_P}{\mathcal{B}_P} \\ &= \mu_0 c \\ &= Z_0. \end{aligned} \tag{9.48}$$

So the standard vacuum impedance relation is recovered at the Planck field scale.

Dimensional check. The electric field expression has dimensions

$$\left[\sqrt{\frac{u_{\mathrm{P}}}{\epsilon_0}}\right] = \sqrt{\frac{\mathrm{J\,m^{-3}}}{\mathrm{F\,m^{-1}}}} = \sqrt{\frac{\mathrm{J\,m^{-3}}}{\mathrm{C\,V^{-1}\,m^{-1}}}} = \sqrt{\frac{\mathrm{J\,V}}{\mathrm{C\,m^2}}} = \sqrt{\frac{\mathrm{V^2}}{\mathrm{m^2}}} = \mathrm{V\,m^{-1}}. \tag{9.49}$$

The magnetic field expression has dimensions

$$[\sqrt{\mu_0 u_{\mathrm{P}}}] = \sqrt{(\mathrm{N\,A^{-2}})(\mathrm{J\,m^{-3}})} = \sqrt{(\mathrm{kg\,m\,s^{-2}\,A^{-2}})(\mathrm{kg\,m^{-1}\,s^{-2}})} = \sqrt{\mathrm{kg^2\,s^{-4}\,A^{-2}}} = \mathrm{kg\,s^{-2}\,A^{-1}} = \mathrm{T}. \tag{9.50}$$

The field ratio has the dimension of speed:

$$\left[\frac{\mathcal{E}_P}{\mathcal{B}_P}\right] = \frac{\mathrm{V\,m^{-1}}}{\mathrm{T}} = \mathrm{m\,s^{-1}}. \tag{9.51}$$

The impedance ratio has the correct dimension:

$$\left[\frac{\mathcal{E}_P}{\mathcal{H}_P}\right] = \frac{\mathrm{V\,m^{-1}}}{\mathrm{A\,m^{-1}}} = \Omega. \tag{9.52}$$

Numerical evaluation. Using CODATA 2022 values,

$$u_{\mathrm{P}} = 4.632\,947 \times 10^{113}\,\mathrm{J\,m^{-3}}. \tag{9.53}$$

The corresponding plane-wave Planck electromagnetic field amplitudes are

$$\mathcal{E}_P = \sqrt{\frac{u_{\mathrm{P}}}{\epsilon_0}} = \sqrt{\frac{4.632\,947 \times 10^{113}\,\mathrm{J\,m^{-3}}}{8.854\,188 \times 10^{-12}\,\mathrm{F\,m^{-1}}}} = 2.287\,466 \times 10^{62}\,\mathrm{V\,m^{-1}}, \tag{9.54}$$

$$\mathcal{B}_P = \sqrt{\mu_0 u_{\mathrm{P}}} = \sqrt{(1.256\,637 \times 10^{-6}\,\mathrm{N\,A^{-2}})(4.632\,947 \times 10^{113}\,\mathrm{J\,m^{-3}})} = 7.630\,791 \times 10^{53}\,\mathrm{T}. \tag{9.55}$$

These satisfy

$$\frac{\mathcal{E}_P}{\mathcal{B}_P} = 2.997\,925 \times 10^{8}\,\mathrm{m\,s^{-1}} = c, \tag{9.56}$$

and

$$\frac{\mathcal{E}_P}{\mathcal{H}_P} = Z_0. \tag{9.57}$$

Convention note. If instead one assigns the full Planck energy density to only one field component,

$$u_{\rm P} = \frac{1}{2}\epsilon_0\mathcal{E}^2 \qquad \text{or} \qquad u_{\rm P} = \frac{\mathcal{B}^2}{2\mu_0}, \tag{9.58}$$

then the resulting single-component amplitudes are larger by a factor of $\sqrt{2}$. This is a convention choice and should not be confused with the total energy density of a vacuum plane wave.

9.5 Summary of Electromagnetic Closure

The electromagnetic closure identities used in this section are

$$\alpha = \frac{v_B}{c}, \qquad q_{\rm P}^2 = \frac{\hbar}{Z_{\rm P}} = \hbar Y_{\rm P}, \qquad Z_{\rm P} = \frac{Z_0}{4\pi}. \tag{9.59}$$

The Planck voltage, current, and power scales are

$$V_P = \frac{E_{\rm P}}{q_{\rm P}}, \qquad I_P = \frac{q_{\rm P}}{t_{\rm P}}, \qquad P_{\rm P} = V_P I_P = \frac{\hbar}{t_{\rm P}^2}. \tag{9.60}$$

The plane-wave Planck electromagnetic field amplitudes are

$$\mathcal{E}_P = \sqrt{\frac{u_{\rm P}}{\epsilon_0}}, \qquad \mathcal{B}_P = \sqrt{\mu_0 u_{\rm P}}, \qquad \frac{\mathcal{E}_P}{\mathcal{B}_P} = c, \qquad \frac{\mathcal{E}_P}{\mathcal{H}_P} = Z_0. \tag{9.61}$$

Thus the electromagnetic Planck-unit sector closes algebraically once the vacuum impedance Z_0 is specified together with the QLM lattice primitives $\{\hbar, \ell_{\rm P}, t_{\rm P}\}$. The impedance normalization

$$Z_{\rm P} = \frac{Z_0}{4\pi}, \qquad Y_{\rm P} = \frac{1}{Z_{\rm P}},$$

connects reduced action to charge through

$$q_{\rm P}^2 = \frac{\hbar}{Z_{\rm P}}.$$

Full-loop impedance conventions may be introduced by replacing $Z_{\rm P}$ with $Z_P^{(\rm loop)} = 2\pi Z_{\rm P} = Z_0/2$ when complete 2π phase cycles are explicitly being considered.

10 Bohr Radius as a Hydrogenic Consistency Check

The Bohr radius provides another hydrogenic calibration of the QLM reduced-action framework. It is not a new QLM primitive. Rather, it shows that the standard hydrogenic length scale is consistent with the same action identity

$$\hbar = m_e v_B a_0, \tag{10.1}$$

and with the lattice transport identity

$$c = \frac{\ell_{\rm P}}{t_{\rm P}}. \tag{10.2}$$

Defining the Bohr momentum by

$$p_B \equiv m_e v_B, \tag{10.3}$$

and using the hydrogenic relation

$$v_B = \alpha c, \tag{10.4}$$

the Bohr radius may be written equivalently as

$$a_0 = \frac{\hbar}{m_e \alpha c} = \frac{\hbar}{m_e v_B} = \frac{\hbar}{p_B}. \tag{10.5}$$

Thus a_0 is the spatial closure scale associated with the Bohr momentum p_B.

QLM-equivalent forms. From the saturated one-tick relation

$$E_P t_P = \hbar, \qquad E_P = \frac{\hbar}{t_P}, \tag{10.6}$$

Eq. (10.5) may be written as

$$a_0 = \frac{E_P t_P}{m_e \alpha c}. \tag{10.7}$$

Using $c = \ell_P / t_P$, this becomes

$$\begin{aligned} a_0 &= \frac{E_P t_P}{m_e \alpha (\ell_P / t_P)} \\ &= \frac{E_P t_P^2}{m_e \alpha \ell_P}. \end{aligned} \tag{10.8}$$

Equations (10.5), (10.7), and (10.8) are algebraically identical. The latter two are QLM substitution forms of the standard hydrogenic expression, not new definitions of the Bohr radius.

Consistency with $\alpha = v_B/c$. The Bohr action identity

$$\hbar = m_e v_B a_0 \tag{10.9}$$

inserted into the first form of Eq. (10.5) gives

$$\begin{aligned} a_0 &= \frac{m_e v_B a_0}{m_e \alpha c} \\ &= a_0 \frac{v_B}{\alpha c}. \end{aligned} \tag{10.10}$$

Canceling a_0 gives

$$1 = \frac{v_B}{\alpha c}, \qquad \alpha = \frac{v_B}{c}. \tag{10.11}$$

This confirms that the Bohr-radius identity, the Bohr action identity, and the fine-structure velocity ratio are mutually consistent.

In this form, the Bohr radius contains no explicit e, ϵ_0, or μ_0. The electromagnetic coupling is encoded through the dimensionless factor α, whose impedance and charge forms were given in Sec. 8.3.

Dimensional check. The standard hydrogenic form has dimensions

$$\begin{aligned} \left[\frac{\hbar}{m_e \alpha c}\right] &= \frac{\mathrm{kg\,m^2\,s^{-1}}}{(\mathrm{kg})(1)(\mathrm{m\,s^{-1}})} \\ &= \mathrm{m}. \end{aligned} \tag{10.12}$$

The momentum form has the same dimension,

$$\begin{aligned} \left[\frac{\hbar}{p_B}\right] &= \frac{\mathrm{kg\,m^2\,s^{-1}}}{\mathrm{kg\,m\,s^{-1}}} \\ &= \mathrm{m}. \end{aligned} \tag{10.13}$$

The QLM-substitution form also has dimension length:

$$\begin{aligned} \left[\frac{E_P t_P^2}{m_e \alpha \ell_P}\right] &= \frac{(\mathrm{kg\,m^2\,s^{-2}})(\mathrm{s^2})}{(\mathrm{kg})(1)(\mathrm{m})} \\ &= \mathrm{m}. \end{aligned} \tag{10.14}$$

Numerical evaluation. Using CODATA 2022 values,

$$\begin{aligned} a_0 &= \frac{\hbar}{m_e \alpha c} \\ &= \frac{1.054\,572 \times 10^{-34}\,\mathrm{J\,s}}{(9.109\,384 \times 10^{-31}\,\mathrm{kg})(7.297\,353 \times 10^{-3})(2.997\,925 \times 10^{8}\,\mathrm{m\,s^{-1}})} \\ &= 5.291\,772 \times 10^{-11}\,\mathrm{m}. \end{aligned} \tag{10.15}$$

The corresponding Bohr speed is

$$\begin{aligned} v_B &= \alpha c \\ &= (7.297\,353 \times 10^{-3})(2.997\,925 \times 10^{8}\,\mathrm{m\,s^{-1}}) \\ &= 2.187\,691 \times 10^{6}\,\mathrm{m\,s^{-1}}. \end{aligned} \tag{10.16}$$

Substituting the QLM identities $E_\mathrm{P} = \hbar/t_\mathrm{P}$ and $c = \ell_\mathrm{P}/t_\mathrm{P}$ reproduces the same Bohr radius identically.

Summary. The Bohr-radius calibration identities are

$$a_0 = \frac{\hbar}{m_e \alpha c} = \frac{\hbar}{m_e v_B} = \frac{\hbar}{p_B} = \frac{E_\mathrm{P} t_\mathrm{P}}{m_e \alpha c} = \frac{E_\mathrm{P} t_\mathrm{P}^2}{m_e \alpha \ell_\mathrm{P}}. \tag{10.17}$$

These forms show that the Bohr radius is a hydrogenic closure scale consistent with the QLM reduced-action and lattice-transport identities.

11 Elementary Charge Closure

The elementary charge is not introduced as an independent QLM primitive. It is obtained from the Planck charge scale through the fine-structure scaling relation. From the per-radian impedance normalization,

$$Z_\mathrm{P} = \frac{Z_0}{4\pi}, \qquad Y_\mathrm{P} = \frac{1}{Z_\mathrm{P}}, \tag{11.1}$$

the QLM Planck charge identity is

$$q_\mathrm{P}^2 = \frac{\hbar}{Z_\mathrm{P}} = \hbar Y_\mathrm{P}. \tag{11.2}$$

This identity is equivalent to the usual SI Planck-charge form. Using

$$\epsilon_0 = \frac{t_\mathrm{P}}{Z_0 \ell_\mathrm{P}}, \qquad c = \frac{\ell_\mathrm{P}}{t_\mathrm{P}}, \tag{11.3}$$

we obtain

$$\begin{aligned} q_\mathrm{P}^2 &= 4\pi\epsilon_0 \hbar c \\ &= 4\pi\hbar \left(\frac{t_\mathrm{P}}{Z_0 \ell_\mathrm{P}}\right)\left(\frac{\ell_\mathrm{P}}{t_\mathrm{P}}\right) \\ &= \frac{4\pi\hbar}{Z_0} \\ &= \frac{\hbar}{Z_\mathrm{P}}. \end{aligned} \tag{11.4}$$

Elementary charge from fine-structure scaling. The fine-structure constant relates the elementary charge to the Planck charge by

$$\alpha = \frac{e^2}{q_{\mathrm{P}}^2}. \tag{11.5}$$

Therefore

$$e^2 = \alpha q_{\mathrm{P}}^2 = \alpha \frac{\hbar}{Z_{\mathrm{P}}} = \alpha \hbar Y_{\mathrm{P}}. \tag{11.6}$$

Equivalently, using $q_{\mathrm{P}}^2 = 4\pi\epsilon_0\hbar c$,

$$e^2 = 4\pi\epsilon_0\hbar c\,\alpha, \tag{11.7}$$

which is the standard rearranged SI form of the fine-structure constant. Thus

$$e = q_{\mathrm{P}}\sqrt{\alpha}. \tag{11.8}$$

The impedance form of the fine-structure constant is

$$\alpha = \frac{e^2 Z_{\mathrm{P}}}{\hbar} = \frac{e^2}{\hbar Y_{\mathrm{P}}}. \tag{11.9}$$

If the loop-aggregated impedance convention is used,

$$Z_P^{(\mathrm{loop})} = 2\pi Z_{\mathrm{P}}, \tag{11.10}$$

then the same relation may be written as

$$\alpha = \frac{e^2 Z_P^{(\mathrm{loop})}}{2\pi\hbar}. \tag{11.11}$$

Dimensional check. The Planck charge identity has dimensions

$$\begin{aligned}\left[\frac{\hbar}{Z_{\mathrm{P}}}\right] &= \frac{\mathrm{J\,s}}{\Omega} \\ &= \frac{\mathrm{V\,C\,s}}{\mathrm{V\,A^{-1}}} \\ &= \mathrm{C\,s\,A} \\ &= \mathrm{C}^2.\end{aligned} \tag{11.12}$$

Therefore $q_{\mathrm{P}}^2 = \hbar/Z_{\mathrm{P}}$ has the correct dimension of charge squared.

The elementary-charge relation has the same dimension because α is dimensionless:

$$[e^2] = [\alpha q_{\mathrm{P}}^2] = \mathrm{C}^2. \tag{11.13}$$

The impedance form of α is dimensionless:

$$\begin{aligned}\left[\frac{e^2 Z_{\mathrm{P}}}{\hbar}\right] &= \frac{(\mathrm{C}^2)(\Omega)}{\mathrm{J\,s}} \\ &= \frac{(\mathrm{C}^2)(\mathrm{V\,A^{-1}})}{\mathrm{J\,s}} \\ &= \frac{(\mathrm{C}^2)(\mathrm{V\,s\,C^{-1}})}{\mathrm{J\,s}} \\ &= \frac{\mathrm{C\,V\,s}}{\mathrm{J\,s}} = 1.\end{aligned} \tag{11.14}$$

Numerical validation. Using CODATA 2022 values,

$$
\begin{aligned}
e^2 &= \alpha \frac{\hbar}{Z_{\mathrm{P}}} \\
&= \alpha\hbar \left(\frac{4\pi}{Z_0}\right) \\
&= (7.297\,353 \times 10^{-3})(1.054\,572 \times 10^{-34}\,\mathrm{J\,s}) \left(\frac{4\pi}{3.767\,303 \times 10^{2}\,\Omega}\right) \\
&= 2.566\,970 \times 10^{-38}\,\mathrm{C}^2.
\end{aligned}
\tag{11.15}
$$

The standard SI route gives the same value:

$$
\begin{aligned}
e^2 &= 4\pi\epsilon_0\hbar c\,\alpha \\
&= 4\pi(8.854\,188 \times 10^{-12}\,\mathrm{F\,m^{-1}})(1.054\,572 \times 10^{-34}\,\mathrm{J\,s})(2.997\,925 \times 10^{8}\,\mathrm{m\,s^{-1}})(7.297\,353 \times 10^{-3}) \\
&= 2.566\,970 \times 10^{-38}\,\mathrm{C}^2.
\end{aligned}
\tag{11.16}
$$

The exact SI elementary charge gives

$$
\begin{aligned}
e^2_{\mathrm{exact}} &= \left(1.602\,177 \times 10^{-19}\,\mathrm{C}\right)^2 \\
&= 2.566\,970 \times 10^{-38}\,\mathrm{C}^2.
\end{aligned}
\tag{11.17}
$$

The impedance and SI routes therefore agree with the exact SI value to the displayed precision. Remaining differences are due only to rounding of the displayed constants; the empirical input is the measured fine-structure constant α.

Summary. The elementary-charge closure identities are

$$
q_{\mathrm{P}}^2 = \frac{\hbar}{Z_{\mathrm{P}}} = \hbar Y_{\mathrm{P}}, \qquad \alpha = \frac{e^2}{q_{\mathrm{P}}^2} = \frac{e^2 Z_{\mathrm{P}}}{\hbar}, \qquad e = q_{\mathrm{P}}\sqrt{\alpha}. \tag{11.18}
$$

Thus the elementary charge is obtained as the fine-structure-scaled fraction of the Planck charge within the QLM per-radian impedance normalization.

12 Conclusion

This paper has developed the Planck-unit foundation of the Quantum Lattice Model (QLM) from the reduced-action phase–flow law

$$
E = \hbar\frac{d\theta}{d\tau}, \qquad c = \frac{\ell_{\mathrm{P}}}{t_{\mathrm{P}}}. \tag{12.1}
$$

For one saturated coherent lattice tick,

$$
\Delta\theta = 1, \qquad \Delta\tau = t_{\mathrm{P}}, \qquad \Delta S = \hbar, \tag{12.2}
$$

the phase–flow law gives

$$
E_{\mathrm{P}} = \frac{\hbar}{t_{\mathrm{P}}}, \qquad E_{\mathrm{P}} t_{\mathrm{P}} = \hbar. \tag{12.3}
$$

Thus the Planck energy is the one-tick reduced-action throughput of the lattice.

With $\{\hbar, \ell_{\rm P}, t_{\rm P}\}$ taken as the primitive QLM lattice set, the standard Planck mechanical quantities reduce algebraically to minimal forms:

$$E_{\rm P} = \frac{\hbar}{t_{\rm P}}, \qquad p_{\rm P} = \frac{\hbar}{\ell_{\rm P}}, \qquad m_{\rm P} = \frac{\hbar t_{\rm P}}{\ell_{\rm P}^2}, \qquad F_{\rm P} = \frac{\hbar}{t_{\rm P}\ell_{\rm P}}, \qquad P_{\rm P} = \frac{\hbar}{t_{\rm P}^2}. \tag{12.4}$$

The associated wave and density scales close in the same way:

$$\omega_P = \frac{1}{t_{\rm P}}, \qquad k_P = \frac{1}{\ell_{\rm P}}, \qquad u_{\rm P} = \frac{\hbar}{\ell_{\rm P}^3 t_{\rm P}}, \qquad \rho_{\rm P} = \frac{\hbar t_{\rm P}}{\ell_{\rm P}^5}. \tag{12.5}$$

The conventional square-root combinations involving $\{\hbar, G, c\}$ are therefore recovered as derived identities rather than introduced as primitive definitions.

The gravitational constant likewise reduces to the lattice form

$$G = \frac{\ell_{\rm P}^5}{t_{\rm P}^3 \hbar}, \tag{12.6}$$

and the exterior Schwarzschild redshift factor can be written equivalently as

$$Z_g(r) = \left(1 - \frac{2GM}{c^2 r}\right)^{-1/2} = \left[1 - 2\left(\frac{M}{m_{\rm P}}\right)\left(\frac{\ell_{\rm P}}{r}\right)\right]^{-1/2}. \tag{12.7}$$

This establishes algebraic compatibility between the QLM Planck-lattice representation and the standard exterior Schwarzschild form.

The electromagnetic Planck-unit sector closes through the per-radian impedance normalization

$$Z_{\rm P} = \frac{Z_0}{4\pi}, \qquad Y_{\rm P} = \frac{1}{Z_{\rm P}}, \qquad q_{\rm P}^2 = \frac{\hbar}{Z_{\rm P}}. \tag{12.8}$$

The corresponding voltage, current, and power scales satisfy

$$V_P = \frac{E_{\rm P}}{q_{\rm P}}, \qquad I_P = \frac{q_{\rm P}}{t_{\rm P}}, \qquad P_{\rm P} = V_P I_P = \frac{\hbar}{t_{\rm P}^2}. \tag{12.9}$$

Vacuum permittivity, permeability, and impedance remain SI electromagnetic response constants, but they are compatible with the QLM transport identity through

$$Z_0 = \mu_0 c = \frac{1}{\epsilon_0 c}, \qquad c = \frac{\ell_{\rm P}}{t_{\rm P}}. \tag{12.10}$$

Hydrogenic structure provides an empirical calibration bridge for the reduced-action framework. The Bohr identities

$$\hbar = m_e v_B a_0, \qquad \alpha = \frac{v_B}{c}, \tag{12.11}$$

show that the same reduced-action quantum governing Planck-scale normalization also appears in atomic orbital closure. In this role, α is a dimensionless calibration ratio connecting hydrogenic orbital motion with electromagnetic coupling and the invariant lattice transport speed. The elementary charge then closes as

$$e^2 = \alpha q_{\rm P}^2, \qquad e = q_{\rm P}\sqrt{\alpha}. \tag{12.12}$$

Across the mechanical, gravitational, electromagnetic, density, thermal, and hydrogenic calibration sectors, the Planck-unit system reduces to a single algebra generated by the QLM primitive triplet $\{\hbar, \ell_{\rm P}, t_{\rm P}\}$, together with sector conversion constants such as k_B and Z_0 and measured dimensionless ratios such as α. No additional dimensional primitives are required at the level of Planck-unit construction.

This work therefore establishes the reduced-action Planck-unit reference layer of the Quantum Lattice Model. Subsequent papers in the QLM series use this layer to develop density saturation and gravitational core structure, Lorentz kinematics from proper ticks, routing admittance, and quantum phase transport.

Notation and Symbol Definitions

Symbol / Notation	Definition
QLM primitive quantities and core identities	
$\hbar$	Reduced Planck constant; primitive reduced-action quantum of the QLM.
ℓ_{P}	Planck length; primitive lattice spatial increment.
t_{P}	Planck time; primitive lattice temporal increment, or one lattice tick.
c	Invariant lattice transport speed, $c = \ell_{\mathrm{P}}/t_{\mathrm{P}}$.
h	Full-cycle Planck constant, $h = 2\pi\hbar$; derived from the per-radian action quantum.
$E = \hbar\, d\theta/d\tau$	QLM phase–flow law defining energy as reduced-action flow per unit proper time.
Phase, action, and wave variables	
θ	Dimensionless phase, measured in radians.
τ	Proper time along a worldline.
S	Action; for one saturated tick, $\Delta S = \hbar$.
ω	Angular frequency, $\omega = d\theta/d\tau$.
ν	Cycle frequency, $\nu = \omega/(2\pi)$.
k	Angular wavenumber, measured per radian.
ω_P	Planck angular frequency, $\omega_P = 1/t_{\mathrm{P}}$.
k_P	Planck angular wavenumber, $k_P = 1/\ell_{\mathrm{P}}$.
Mechanical and thermodynamic Planck quantities	
E_{P}	Planck energy, $E_{\mathrm{P}} = \hbar/t_{\mathrm{P}}$.
p_{P}	Planck momentum, $p_{\mathrm{P}} = \hbar/\ell_{\mathrm{P}} = \hbar k_P$.
m_{P}	Planck mass, $m_{\mathrm{P}} = \hbar t_{\mathrm{P}}/\ell_{\mathrm{P}}^2$.
F_{P}	Planck force, $F_{\mathrm{P}} = \hbar/(t_{\mathrm{P}}\ell_{\mathrm{P}})$.
a_{P}	Planck acceleration, $a_{\mathrm{P}} = \ell_{\mathrm{P}}/t_{\mathrm{P}}^2 = c/t_{\mathrm{P}}$.
P_{P}	Planck power, $P_{\mathrm{P}} = \hbar/t_{\mathrm{P}}^2$.
T_{P}	Planck temperature, $T_{\mathrm{P}} = \hbar/(t_{\mathrm{P}} k_B)$.
k_B	Boltzmann constant; thermal conversion constant, not a QLM lattice primitive.
Geometric cells and densities	
A_P	Planck area, $A_P = \ell_{\mathrm{P}}^2$.
$\mathcal{V}_P$	Planck spatial volume, $\mathcal{V}_P = \ell_{\mathrm{P}}^3$.
V_4	Planck four-volume, $V_4 = \ell_{\mathrm{P}}^3 t_{\mathrm{P}}$.
Φ_P	Planck phase-density scale, $\Phi_P = 1/(\ell_{\mathrm{P}}^3 t_{\mathrm{P}})$.
u_{P}	Planck energy density, $u_{\mathrm{P}} = \hbar/(\ell_{\mathrm{P}}^3 t_{\mathrm{P}}) = \hbar\Phi_P$.
ρ_{P}	Planck mass density, $\rho_{\mathrm{P}} = m_{\mathrm{P}}/\ell_{\mathrm{P}}^3 = \hbar t_{\mathrm{P}}/\ell_{\mathrm{P}}^5$.
Gravity notation	
G	Newtonian gravitational constant; QLM lattice form $G = \ell_{\mathrm{P}}^5/(t_{\mathrm{P}}^3\hbar)$.
M	Gravitating mass in Schwarzschild and redshift relations.
r	Radial coordinate or distance from the gravitating mass.
r_s	Schwarzschild radius, $r_s = 2GM/c^2 = 2(M/m_{\mathrm{P}})\ell_{\mathrm{P}}$.

Symbol / Notation	Definition
$Z_g(r)$	Gravitational impedance/redshift factor, $Z_g(r) = \big(1 - 2GM/(c^2 r)\big)^{-1/2}$.
Electromagnetic notation	
e	Elementary charge.
ϵ_0	Vacuum permittivity.
μ_0	Vacuum permeability.
Z_0	Vacuum impedance, $Z_0 = \mu_0 c = 1/(\epsilon_0 c)$.
q_{P}	Planck charge, $q_{\mathrm{P}}^2 = 4\pi\epsilon_0\hbar c = \hbar/Z_{\mathrm{P}}$.
Z_{P}	Per-radian Planck impedance, $Z_{\mathrm{P}} = Z_0/(4\pi)$.
Y_{P}	Per-radian Planck admittance, $Y_{\mathrm{P}} = 1/Z_{\mathrm{P}}$.
$Z_P^{(\mathrm{loop})}$	Full-cycle impedance convention, $Z_P^{(\mathrm{loop})} = 2\pi Z_{\mathrm{P}} = Z_0/2$, used only when full 2π phase loops are explicitly discussed.
V_P	Planck voltage, $V_P = E_{\mathrm{P}}/q_{\mathrm{P}}$.
I_P	Planck current, $I_P = q_{\mathrm{P}}/t_{\mathrm{P}}$.
$\mathcal{E}_P$	Plane-wave Planck electric field amplitude, $\mathcal{E}_P = \sqrt{u_{\mathrm{P}}/\epsilon_0}$.
$\mathcal{B}_P$	Plane-wave Planck magnetic flux density, $\mathcal{B}_P = \sqrt{\mu_0 u_{\mathrm{P}}}$.
$\mathcal{H}_P$	Planck magnetic field intensity, $\mathcal{H}_P = \mathcal{B}_P/\mu_0$.
Hydrogenic and atomic calibration quantities	
m_e	Electron mass.
v_B	Bohr orbital speed in the hydrogen ground state.
a_0	Bohr radius.
p_B	Bohr momentum, $p_B = m_e v_B$.
E_B	Bohr kinetic energy, $E_B = \frac{1}{2} m_e v_B^2$.
α	Fine-structure constant; calibration identities include $\alpha = v_B/c = e^2 Z_{\mathrm{P}}/\hbar = e^2/q_{\mathrm{P}}^2$.
R_∞	Rydberg constant.
k_∞	Angular Rydberg wavenumber, $k_\infty = 2\pi R_\infty$.
ν_R	Rydberg frequency, $\nu_R = cR_\infty$.
ω_R	Angular Rydberg frequency, $\omega_R = 2\pi\nu_R = E_B/\hbar$.
Conventions	
per-radian	Normalized to one radian of phase advance. This is the default QLM convention.
full-cycle / loop	Normalized over a complete 2π phase cycle. Used only when explicitly stated.
primitive	Taken as foundational in the QLM lattice construction. In this paper the primitive set is $\{\hbar, \ell_{\mathrm{P}}, t_{\mathrm{P}}\}$.
derived	Obtained algebraically from the primitive set and sector conversion constants.

Fundamental Constants and QLM Planck-Unit Reference Values

Table 3: Reference constants and Planck-unit values used in this paper. Exact SI constants are fixed by definition. CODATA 2022 values are used for measured constants. Within QLM, the primitive lattice set is $\{\hbar, \ell_{\rm P}, t_{\rm P}\}$; quantities such as G, Z_0, k_B, and α enter as conversion, response, or calibration quantities rather than additional lattice primitives.

Quantity	**Symbol**	**Standard / SI Form**	**Value / Status**
		Exact SI-defined constants	
Speed of light	c	defined	$2.997\,925 \times 10^{8}\,\mathrm{m\,s^{-1}}$ exact
Planck constant	h	defined	$6.626\,070 \times 10^{-34}\,\mathrm{J\,s}$ exact
Reduced Planck constant	$\hbar$	$h/(2\pi)$	exact, derived from h
Elementary charge	e	defined	$1.602\,177 \times 10^{-19}\,\mathrm{C}$ exact
Boltzmann constant	k_B	defined	$1.380\,649 \times 10^{-23}\,\mathrm{J\,K^{-1}}$ exact
		CODATA 2022 measured or inferred constants	
Gravitational constant	G	measured	$6.674\,300 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$
Fine-structure constant	α	measured dimensionless ratio	$7.297\,353 \times 10^{-3}$
Vacuum impedance	Z_0	$\mu_0 c$	$3.767\,303 \times 10^{2}\,\Omega$
Vacuum permittivity	ϵ_0	$1/(Z_0 c)$	$8.854\,188 \times 10^{-12}\,\mathrm{F\,m^{-1}}$
Vacuum permeability	μ_0	Z_0/c	$1.256\,637 \times 10^{-6}\,\mathrm{N\,A^{-2}}$
		Standard reduced-action Planck definitions	
Planck time	$t_{\rm P}$	$\sqrt{\hbar G/c^5}$	$5.391\,247 \times 10^{-44}\,\mathrm{s}$
Planck length	$\ell_{\rm P}$	$\sqrt{\hbar G/c^3}$	$1.616\,255 \times 10^{-35}\,\mathrm{m}$
Planck mass	$m_{\rm P}$	$\sqrt{\hbar c/G}$	$2.176\,434 \times 10^{-8}\,\mathrm{kg}$
Planck energy	$E_{\rm P}$	$\sqrt{\hbar c^5/G}$	$1.956\,081 \times 10^{9}\,\mathrm{J}$
Planck charge	$q_{\rm P}$	$\sqrt{4\pi\epsilon_0\hbar c}$	$1.875\,546 \times 10^{-18}\,\mathrm{C}$

QLM lattice relations. Within the Quantum Lattice Model, the same Planck system is generated from the primitive lattice set $\{\hbar, \ell_{\rm P}, t_{\rm P}\}$ and the transport identity $c = \ell_{\rm P}/t_{\rm P}$:

$$
\begin{aligned}
E_{\rm P} &= \frac{\hbar}{t_{\rm P}}, & p_{\rm P} &= \frac{\hbar}{\ell_{\rm P}}, \\
m_{\rm P} &= \frac{\hbar t_{\rm P}}{\ell_{\rm P}^2}, & F_{\rm P} &= \frac{\hbar}{t_{\rm P}\ell_{\rm P}}, \\
P_{\rm P} &= \frac{\hbar}{t_{\rm P}^2}, & G &= \frac{\ell_{\rm P}^5}{t_{\rm P}^3\hbar}, \\
q_{\rm P}^2 &= \frac{\hbar}{Z_{\rm P}}, & Z_{\rm P} &= \frac{Z_0}{4\pi}.
\end{aligned}
\tag{12.13}
$$

Thus the conventional reduced-action Planck-unit construction based on $\{\hbar, G, c\}$ and the QLM lattice construction based on $\{\hbar, \ell_{\rm P}, t_{\rm P}\}$ are algebraically equivalent, while the QLM ontology takes $\{\hbar, \ell_{\rm P}, t_{\rm P}\}$ as primitive.

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